TFY4305 solutions exercise set 1 2014

Problem 2.2.3

The equation reads

$$\dot{x} \;=\; x(1-x^2)\;, \tag{1}$$

The fixed points are the solutions to $x(1-x^2)=0$. This yields

$$\underline{\underline{x=0\;, \qquad x=\pm 1}}\;. \tag{2}$$

Furthermore, $f'(x) = 1-3x^2$ Since $f'(x=0)=1$, $x=0$ is an $\underline{\underline{\text{unstable fixed point}}}$. Since $f'(x=\pm 1) = -2$, $x=\pm 1$ are $\underline{\underline{\text{stable fixed points}}}$.

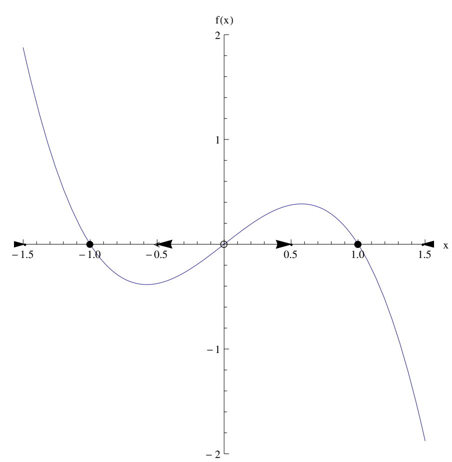

Figure 1: Vector field including the three fixed points.

The exact solution can be found by separation of variables.

$$\frac{dx}{x(1-x^2)} \;=\; dt\;, \tag{3}$$

The left-hand side can be rewritten using partial fractional decomposition.

$$\int dx\left[\frac{1}{x}-\frac{1}{2}\frac{1}{x-1}-\frac{1}{2}\frac{1}{x+1}\right] \;=\; \int dt\;. \tag{4}$$

Integrating term by term and rearranging yields

$$\frac{1}{2}\ln\left|\frac{x^2}{x^2-1}\right| \quad = \quad t + C\ , \tag{5}$$

where C is an integration constant. Exponentiating gives

$$\pm\frac{x^2}{x^2-1} \quad = \quad Ke^{2t}\ , \tag{6}$$

where $K = e^{2C}$. This is quadratic equation for x and can be easily solved. For the upper sign, we obtain

$$x(t) \quad = \quad \underline{\underline{\pm\frac{\sqrt{K}e^t}{\sqrt{Ke^{2t}-1}}}}\ . \tag{7}$$

For the lower sign, we find

$$x(t) \quad = \quad \underline{\underline{\pm\frac{\sqrt{K}e^t}{\sqrt{Ke^{2t}+1}}}}\ . \tag{8}$$

Note that the solutions tend to $x = \pm 1$ as $t \to \infty$. In the first case, we have $x(0) = \pm\sqrt{\frac{K}{K-1}}$ and corresponds to the initial condition where $|x(0)| > 1$. In the second case, we have $x(0) = \pm\sqrt{\frac{K}{K+1}}$ and corresponds to the initial condition where $|x(0)| < 1$. If the initial condition is $x(0) = \pm 1$, we are at a stable fixed point and we have $x(t) = \pm 1$ for all t. Finally if $x(0) = 0$, we start at the unstable fixed point and remain there. Also note the inflection points at $x = \pm\frac{1}{\sqrt{3}}$.

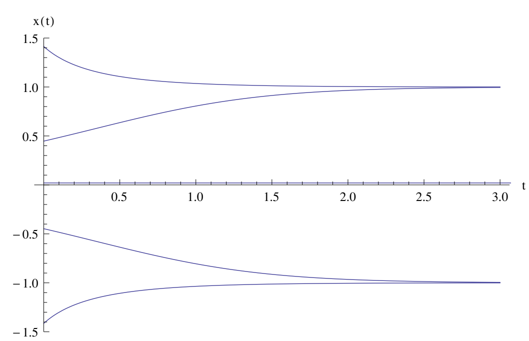

Figure 2: Exact solution for various initial values as explained in main text.

Problem 2.4.7

We have

$$\dot{x} \quad = \quad ax - x^3\ . \tag{9}$$

The fixed points are found by solving $ax - x^3$. $x = 0$ is always a fixed point and $x = \pm\sqrt{a}$ for $a > 0$. Moreover,

$$f'(x) = a - 3x^2 , \tag{10}$$

and therefore $f(x = 0) = a$. $x = 0$ is thus unstable for $a > 0$ and stable for $a < 0$. For $a = 0$, we have $f(x) = -x^3$ which is negative for positive x and vice versa. Hence $x = 0$ is a stable fixed point for $a = 0$. For $x = \pm\sqrt{a}$, we have $f'(\pm\sqrt{a}) = -2a$ which is negative for positive a and hence these fixed points are stable.

Problem 2.6.1

The point is that the harmonic oscillator is not a first-order system. It is a system of two coupled differential equations. Define $\dot{x} = y$. This yields

$$\begin{aligned} m\dot{y} &= -kx \quad (11) \\ \dot{x} &= y , \quad (12) \end{aligned}$$

and we conclude that the system is two dimensional and so does not correspond to flow on the line.

Problem 2.7.6

The dynamics is governed by the equation

$$\dot{x} = r + x - x^3 , \tag{13}$$

where r is a parameter. The potential function $V(x)$ is given by integrating $V'(x) = -f(x) = -r - x + x^3$. This yields

$$V(x) = \underline{\underline{-rx - \frac{1}{2}x^2 + \frac{1}{4}x^4 + C}} , \tag{14}$$

where C is an integration constant which we henceforth set to zero.

In order to gain insight into the number and position of fixed points as a function of the parameter r, it is useful to plot the function $g(x) = x^3 - x$ and the horizontal line $h(x) = r$. The fixed points are then given by the solutions to $g(x) = h(x)$.

The extrema $x_\pm$ of $g(x)$ are given by

$$\begin{aligned} g'(x) &= 3x^2 - 1 \\ &= 0 , \quad (15) \end{aligned}$$

which gives $x_\pm = \pm 1/\sqrt{3}$. Thus $|g(x_\pm)| = 2/3\sqrt{3}$. This implies that there is one fixed point for $|r| > 2/3\sqrt{3}$, two fixed points for $|r| = 2/3\sqrt{3}$ and three fixed points for $|r| < 2/3\sqrt{3}$. This is shown in Fig. 3.

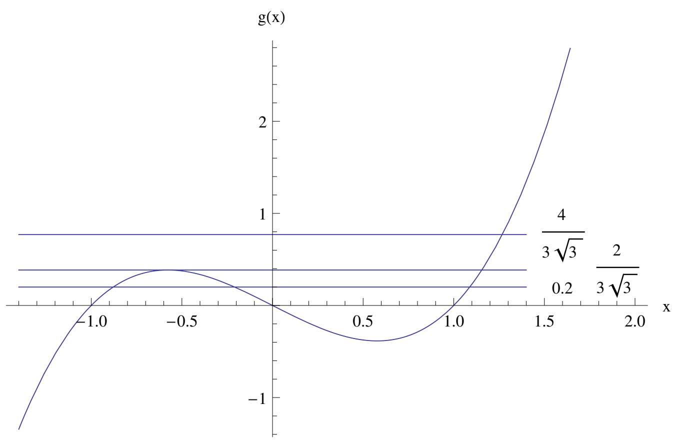

Figure 3: Plot of the functions $g(x)$ and the horizontal line $h(x) = r$ for various values of r.

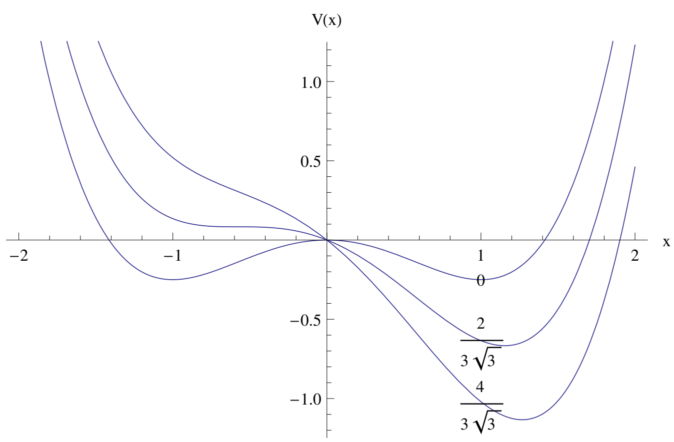

Figure 4: Plot of the potential $V(x)$ for various values of r.

The potential $V(x)$ is plotted for the same values of r in Fig. 4. For $r = 0$, we see that there are two minima, namely $x = \pm 1$ and one maximum $x = 0$. These correspond to two stable fixed points and one unstable fixed point. For $r = 2/3\sqrt{3}$, we see that the stable fixed point to the left of the origin has merged with the unstable minimum and is half-stable. For $r = 4/3\sqrt{3}$, there is only stable fixed point to the right of the origin.