TFY4305 solutions exercise set 10 2014

Problem 6.5.12

The dynamics is governed by the set of equations

$$\begin{aligned} \dot{x} &= xy \, , & (1) \\ \dot{y} &= -x^2 \, . & (2) \end{aligned}$$

a) One finds

$$\begin{aligned} \frac{dE}{dt} &= \frac{d}{dt}\left(x^2 + y^2\right) \\ &= 2(x\dot{x} + y\dot{y}) \\ &= \underline{\underline{0}} \, , & (3) \end{aligned}$$

where we in the last line have used the expressions for $\dot{x}$ and $\dot{y}$. This shows that $E =$ constant.

b) The fixed points are given by $f(x, y) = g(x, y) = 0$, where $f(x, y) = xy$ and $g(x, y) = -x^2$. The second equation gives $x = 0$ and for this value the first equation is also satisfied. Hence there is a $\underline{\underline{\text{line of fixed points given by } x = 0}}$, i.e. the y-axis.

c) Since $dE/dt = 2rdr/dt = 0$, the radius of a trajectory is constant. Thus the flow is on the circle. Moreover

$$\begin{aligned} \dot{\theta} &= \frac{x\dot{y} - y\dot{x}}{r^2} \\ &= \frac{-x^3 - xy^2}{r^2} \\ &= -r\cos\theta \, . & (4) \end{aligned}$$

This shows that a circular flow has two lines of fixed points, namely $\theta = \frac{1}{2}\pi$ and $\theta = \frac{3}{2}\pi$. This is simply the y-axis. In the upper halfplane, $\dot{x} > 0$ for $x > 0$ and $\dot{x} < 0$ for $x < 0$, and vice versa ($\dot{y} < 0$ everywhere). Hence, we flow away from the y-axis If we are in the lower halfplane, $\dot{x} > 0$ for $x < 0$ and vice versa ($\dot{y} < 0$ everywhere). Hence, we flow toward the y-axis The phase portrait is shown in Fig. 1.

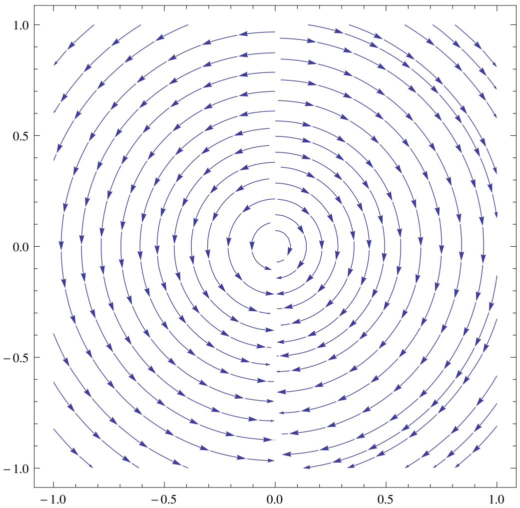

Figure 1: Phase portrait of problem 6.5.12.

Problem 6.7.2

The equation of motion reads

$$\ddot{\theta} + \sin\theta \;=\; \gamma\,. \tag{5}$$

Introducing $\nu = \dot{\theta}$, we can write the equation as

$$\dot{\theta} \;=\; \nu\,, \tag{6}$$

$$\dot{\nu} \;=\; \gamma - \sin\theta\,. \tag{7}$$

We notice that the system is reversible.

a) The fixed points are given by $\nu^* = 0$ and $\sin\theta^* = \gamma$. For $\gamma > 1$, there are no fixed points. For $\gamma = 1$, $(\frac{1}{2}\pi, 0)$ is the only fixed point, and for $\gamma < 1$, there are two fixed points. The solution $0 < \theta_1^* < \frac{1}{2}\pi$ is stable and the solution $\frac{1}{2}\pi < \theta_2^* < \pi$ is unstable. This is a saddle-point bifurcation. The Jacobian matrix reads

$$A(\theta,\nu) \;=\; \begin{pmatrix} 0 & 1 \\ -\cos\theta & 0 \end{pmatrix} \tag{8}$$

The characteristic equation becomes $\lambda^2 + \cos\theta = 0$ and so

$$\lambda \;=\; \pm i\sqrt{\cos\theta_1^*}\,, \tag{9}$$

since $\cos\theta_1^* > 0$. This yields $\Delta = \cos\theta_1^* > 0$ and $\tau = 0$. Thus the fixed point θ_1^* is a center. For the fixed point θ_2^*, we find

$$\lambda \;=\; \pm\sqrt{-\cos\theta_2^*}\,, \tag{10}$$

since $\cos\theta_2^* < 0$. This yields $\Delta = \cos\theta_2^* < 0$ and $\tau = 0$. Thus the fixed point θ_2^* is a saddle point.

c) The equation gives

$$\dot{\theta}(\ddot{\theta} + \sin\theta) \quad = \quad \gamma\dot{\theta}. \tag{11}$$

Integration yields

$$\frac{1}{2}\dot{\theta}^2 - \cos\theta = \gamma\theta + C\ . \tag{12}$$

This equation makes no sense since the right-hand-side is not a single-valued function of θ. Hence, the system is not conservative. The reason is that the external torque is doing work on the system and hence energy is pumped into it.

d) The phase portrait is shown for different values of γ in Fig. 2.

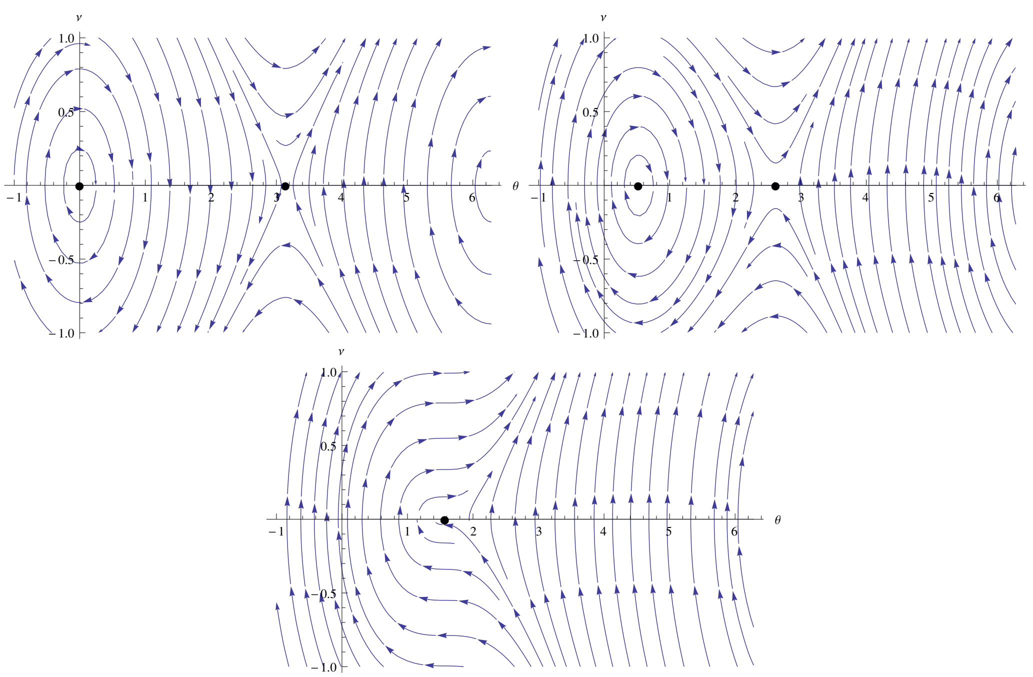

Figure 2: Phase portrait of problem 6.7.2. for $\gamma = 0$, $\gamma = \frac{1}{2}$, and $\gamma = 1$. Fixed points are indicated by the black blobs.

e) For the center, we have found the frequency $\omega = \sqrt{\cos\theta_1^*}$, where $\sin\theta_1^* = \gamma$. This yields $\cos\theta_1^* = (1-\gamma^2)^{\frac{1}{2}}$ and therefore

$$\omega \quad = \quad \underline{\underline{(1-\gamma^2)^{\frac{1}{4}}}}\ . \tag{13}$$