TFY4305 solutions exercise set 13 2014

Problem 7.3.1

The dynamics is given by the equations

$$\begin{aligned} \dot{x} &= x - y - x(x^2 + 5y^2)\ , && (1) \\ \dot{y} &= x + y - y(x^2 + y^2)\ . && (2) \end{aligned}$$

a) The Jacobian matrix is given by

$$A(x, y) = \begin{pmatrix} 1 - 3x^2 - 5y^2 & -1 - 10xy \\ 1 - 2xy & 1 - x^2 - 3y^2 \end{pmatrix} . \quad (3)$$

Evaluated at the origin, we find

$$A(0, 0) = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} . \quad (4)$$

The eigenvalues are given by the equation $(\lambda - 1)^2 + 1 = 0$, i.e.

$$\lambda = 1 \pm i. \quad (5)$$

Hence the origin is an $\underline{\underline{\text{unstable spiral}}}$.

b) We have

$$\begin{aligned} r\dot{r} &= x\dot{x} + y\dot{y} \\ &= x^2 + y^2 - x^2(x^2 + 5y^2) - y^2(x^2 + y^2) \\ &= r^2 - r^4(1 + 4\cos^2\theta \sin^2\theta)\ , && (6) \end{aligned}$$

and so

$$\dot{r} = \underline{\underline{r\left(1 - r^2(1 + 4\cos^2\theta \sin^2\theta)\right)}} . \quad (7)$$

Similarly:

$$\begin{aligned}\dot{\theta} &= \frac{x\dot{y} - y\dot{x}}{r^2} \\ & \frac{x^2 + xy - xy(x^2+y^2) - xy + y^2 + xy(x^2+5y^2)}{r^2} \\ &= \underline{\underline{1 + 4r^2\cos\theta\sin^3\theta}} \ . \end{aligned} \tag{8}$$

c) The condition $\dot{r} > 0$ translates into $1 - r^2(1 + 4\cos^2\theta\sin^2\theta) = 1 - r^2(1+\sin^2 2\theta) > 0$. This is satisfied for all θ if $1 - 2r^2 > 0$ since $\sin^2 2\theta \leq 1$. Thus

$$r_1 = \underline{\underline{\frac{1}{\sqrt{2}}}} \ . \tag{9}$$

d) A similar argument gives $\dot{r} < 0$ if $1 - r^2(1+\sin^2 2\theta) < 0$, and is satisfied if $1 - r^2 < 0$ (Since $\sin^2 2\theta \geq 0$). Thus

$$r_2 = \underline{\underline{1}} \ . \tag{10}$$

e) A fixed point must satisfy $\dot{r} = 0$, i. e. $\left(1 - r^2(1 + 4\cos^2\theta\sin^2\theta)\right) = 0$. Inserting this into the equation $\dot{\theta} = 0$, we obtain

$$1 + 4\cos\theta\sin^2\theta(\sin\theta + \cos\theta) = 0 \ . \tag{11}$$

This equation has no solution (see Fig. 1) and the system has therefore no fixed point.

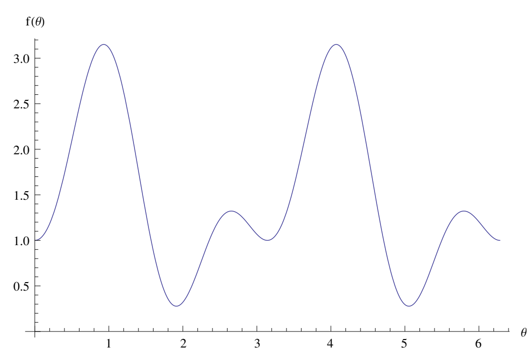

Figure 1: Left-hand side of Eq. (11): $f(\theta) = 1 + 4\cos\theta\sin^2\theta(\sin\theta + \cos\theta)$.

The Poincare-Bendixson theorem then implies that there is a limit cycle within the trapping region given by the annulus with $r_1 = 1/\sqrt{2}$ and $r_1 = 1$. This is shown in Fig. 2

Problem 7.3.4

The dynamics of the equation is given by

$$\begin{aligned}\dot{x} &= x(1 - 4x^2 - y^2) - \frac{1}{2}y(1+x) \ , \qquad (12) \\ \dot{y} &= y(1 - 4x^2 - y^2) + 2x(1+x) \ . \qquad (13)\end{aligned}$$

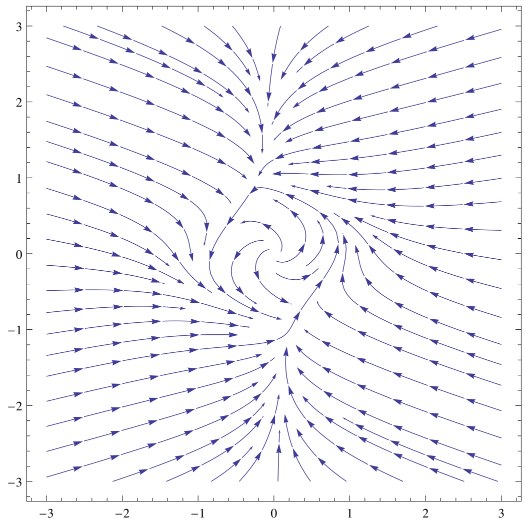

Figure 2: Phase portrait of problem 7.3.1.

a) Clearly the origin is a fixed point. The Jacobian matrix is given by

$$A(x,y) = \begin{pmatrix} 1-y^2-12x^2-\frac{1}{2}y & -2xy-\frac{1}{2}(1+x) \\ -8xy+2+4x & 1-4x^2-3y^2 \end{pmatrix} . \quad (14)$$

Evaluated at the origin, we find

$$A(0,0) = \begin{pmatrix} 1 & -\frac{1}{2} \\ 2 & 1 \end{pmatrix} . \quad (15)$$

The eigenvalues are given by the equation $(\lambda-1)^2+1=0$, i.e. $\lambda = 1 \pm i$ and so the origin is an unstable spiral.

b) Let $V(x,y) = (1-4x^2-y^2)^2$. This yields

$$\begin{aligned} \frac{dV}{dt} &= \frac{\partial V}{\partial x}\dot{x} + \frac{\partial V}{\partial y}\dot{y} \\ &= -4(1-4x^2-y^2)^2(4x^2+y^2) . \end{aligned} \quad (16)$$

For points not on the ellipse $4x^2+y^2=1$, we have $\dot{V}<0$. This tells ut that we flow towards lower values of V. We have $V=0$ on the ellipse $4x^2+y^2=1$ and $V>0$ away from it. Hence we will approach the ellipse as $t \to \infty$.