TFY4305 solutions exercise set 14 2014

Problem 7.2.16

We can use the counterexample

$$\begin{aligned} \dot{r} &= \frac{1}{2} r \log r \ , && (1) \\ \dot{\theta} &= 1 \ . && (2) \end{aligned}$$

Choosing $g = 1$ in Dulac's theorem, we obtain

$$\begin{aligned} \nabla \cdot \dot{\mathbf{x}} &= \frac{1}{2r}\frac{\partial}{\partial r} r^2 \log r + \frac{1}{r}\frac{\partial}{\partial \theta} 1 \\ &= (\log r + 1) \ . && (3) \end{aligned}$$

This expression is positive in the region given by $\log r + 1 > 0$, i. e. for $r > e^{-1/2} \approx 0.606531$. The region $r \geq e^{-1/2}$, $\theta \in (0, 2\pi)$ is not simply connected and so the curve $r^2 = 1/e$ is drawn as a dashed circle. The requirements of the theorem are not satisfied. The phase portrait is shown in Fig. 1. The closed unstable circle $r = 1$ is drawn as a solid circle. .

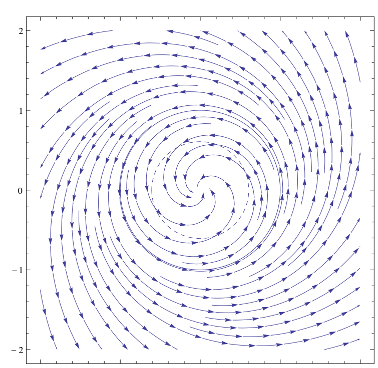

Figure 1: Phase portrait of problem 7.2.16.

Problem 7.3.6

$$\ddot{x} + F(x, \dot{x})\dot{x} + x \;=\; 0\;. \tag{4}$$

a) We have ordinary damping if the distance to the origin in phase space r is larger than b. We have negative damping (energy pumped into the system) if the distance r from the origin in phase is smaller than a.

b) The equation is rewritten as

$$\dot{x} \;=\; y\;, \tag{5}$$

$$\dot{y} \;=\; -F(x, y)y - x\;. \tag{6}$$

This yields

$$\begin{aligned} r\dot{r} &= x\dot{x} + y\dot{y} \\ &= -F(x, y)y^2\;, \end{aligned} \tag{7}$$

which shows that $\dot{r} > 0$ for $r \leq a$ and $\dot{r} < 0$ for $r \geq b$. Hence the annulus between the circles with radii a and b is a trapping region. There are no fixed points inside the annulus and so the Poincare-Bendixon theorem applies.

Problem 7.3.10

The equations can be written as

$$\dot{x} \;=\; ax + by - r^2 x\;, \tag{8}$$

$$\dot{y} \;=\; cx + dy - r^2 y\;, \tag{9}$$

where the matrix A is

$$A(x, y) \;=\; \begin{pmatrix} a & b \\ c & d \end{pmatrix}\;. \tag{10}$$

The origin is a fixed point and the Jacobian matrix evaluated at the origin is the matrix A since the higher-order terms in Eqs. (8) and (9) vanish after linearization. The matrix A is real and has two complex eigenvalues $\alpha \pm i\omega$. If $\alpha > 0$, then we know that the origin is an unstable spiral. We can then find a small neighborhood such that the vector field is pointing away from the origin. If we chose a circle with sufficiently large radius R, the velocity field will be dominated by the terms $-r^2 x$ and $-r^2 y$. On such a circle the flow will be be inwards. By excluding the small neighborhood around the origin from the region inside the circle with radius R, we can apply the Poincare-Bendixon theorem and conclude there exists at least one limit cycle.

On the other hand, if $\alpha < 0$, the origin is a stable spiral. We can apply Dulac's criterion by calculating the divergence of $\nabla \cdot (g(x,y)\boldsymbol{x}(x,y))$, where $g(x,y) = 1$:

$$\begin{aligned} \nabla \cdot \boldsymbol{x}(x,y) &= a - 3x^2 - y^2 + b - x^2 - 3y^2 \\ &= \tau - 4r^2 \ . \end{aligned} \tag{11}$$

Since $\alpha = \frac{1}{2}\tau < 0$, we see that $\nabla \cdot (g(x,y)\boldsymbol{x}(x,y)) < 0$ throughout the plane. Dulac's theorem then guarantees there are no closed orbits.