TFY4305 solutions exercise set 15 2014

Problem 8.2.1

The biased van der Pol oscillator reads

$$\ddot{x} + \mu(x^2 - 1)\dot{x} + x \;=\; a \,. \tag{1}$$

This equation can be rewritten as

$$\dot{x} \;=\; y \,, \tag{2}$$
$$\dot{y} \;=\; -\mu(x^2 - 1)y - x + a \,. \tag{3}$$

The fixed point is $(a, 0)$ and the Jacobian matrix is

$$A(x, y) \;=\; \begin{pmatrix} 0 & 1 \\ -1 - 2\mu xy & -\mu(x^2 - 1) \end{pmatrix} . \tag{4}$$

This yields

$$A(a, 0) \;=\; \begin{pmatrix} 0 & 1 \\ -1 & -\mu(a^2 - 1) \end{pmatrix} , \tag{5}$$

and so the eigenvalues are

$$\lambda \;=\; \frac{-\mu(a^2 - 1) \pm \sqrt{\mu^2(a^2 - 1)^2 - 4}}{2} \tag{6}$$

The real part of the eigenvalue changes sign when $\mu(a^2 - 1) = 0$, i. e. for $a = \pm 1$ or $\mu = 0$ For parameter values close to these values, the imaginary part of the eigenvalue is nonzero. Hence the Hopf bifurcations occur for the lines $\mu = 0$ and $a = \pm 1$ in the (μ, a) plane. Note that the system becomes conservative for $\mu = 0$ [1]. Thus the bifurcation taking place $\mu = 0$ is the degenerate type: The fixed point $(a, 0)$ changes from a stable to an unstable spiral. but there are no closed curves on either side of $\mu = 0$. The conserved quantity for $\mu = 0$ is

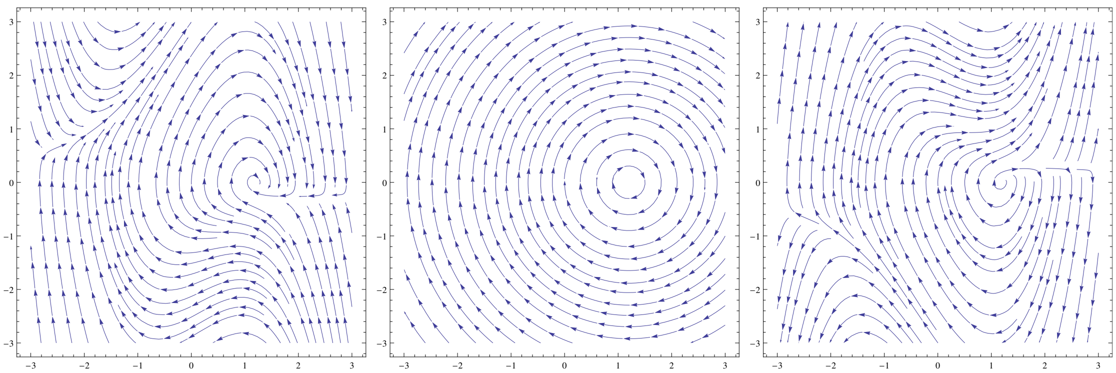

Figure 1: Phase portrait for $a = 1.2$ and $\mu = -1, 0$ and $+1$.

$\frac{1}{2}\dot{x}^2 + \frac{1}{2}x^2 - ax$. Only for $\mu = 0$ is there a continuum of circular trajectories [2]. This is shown in Fig. 1 for $a = 1.2$ and $\mu = 1, 0$, and -1.

In Fig. 2, we plot the phase portrait for $\mu = 1$ and $a = 0.8, 1$ and 1.2. For $a = 0.8$, we have an unstable spiral which disappears as a goes through 1 and becomes a a stable spiral for $a > 1$. At the same time there is a closed orbit for $a \leq 1$ and so we have a supercritical Hopf bifurcation at $a = 1$.

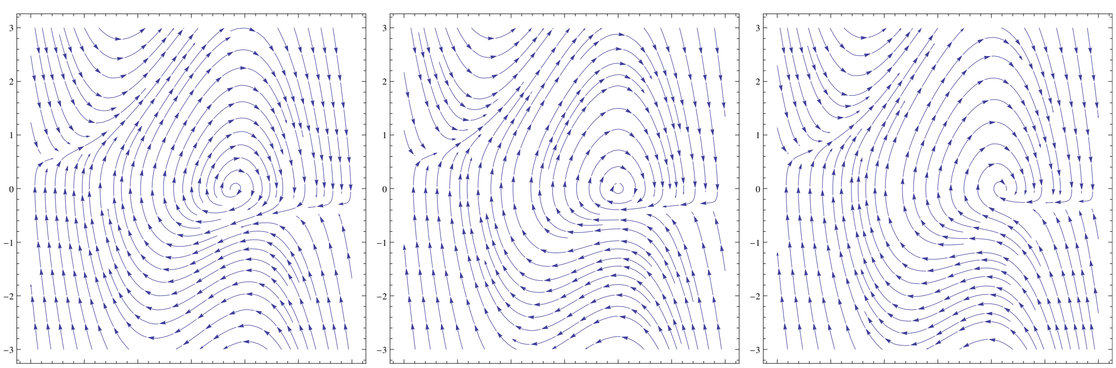

Figure 2: Phase portrait for $\mu = 1$ and $a = 0.8, 1$ and 1.2.

[1]It is also reversible for $\mu = 0$.

[2]Note that $a = \pm 1$ are special points. In this case the eigenvalues are purely imaginary independent of μ. We then have closed curves for all values of μ.