TFY4305 solutions exercise set 17 2014

Problem 10.3.2

The logistic map is given by

$$x_{n+1} = rx_n(1-x_n) \; . \tag{1}$$

a) The superstability is given by

$$\begin{aligned} f[f(x)]' &= f'(p)f'(q) \\ &= r(1-2p)r(1-2q) \\ &\stackrel{!}{=} 0 \; . \end{aligned} \tag{2}$$

Thus either $p=\frac{1}{2}$ or $q=\frac{1}{2}$.

b) The points are given by (see textbook p359)

$$p,q = \frac{r+1\pm\sqrt{(r-3)(r+1)}}{2r} \tag{3}$$

Inserting $p=\frac{1}{2}$ on the left-hand-side, yields

$$r = r+1\pm\sqrt{(r-3)(r+1)} \tag{4}$$

or

$$r^2-2r-4 = 0 \; . \tag{5}$$

The solutions are $r=1\pm\sqrt{5}$ and we must choose the positive solution. This yields $r=\underline{\underline{1+\sqrt{5}}}$.

Problem 10.3.7

a) The decimal shift map is given by

$$x_{n+1} = 10x_n \pmod 1 \; . \tag{6}$$

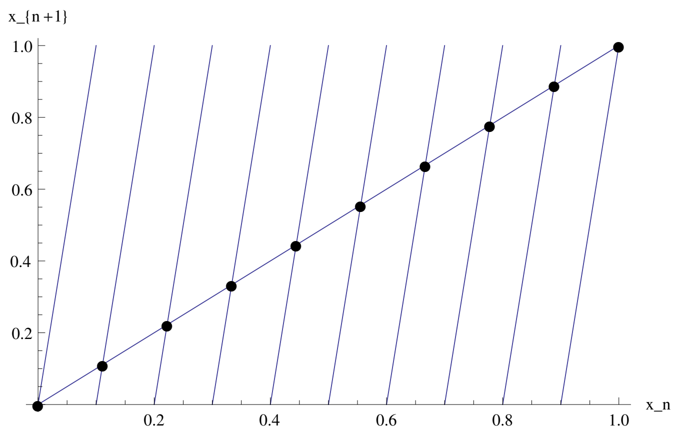

Figure 1: Decimal shift map.

The map is shown is Fig. 1.
b) We can find the fixed points graphically by drawing the line $y = x$ and see where it intersects the map shown in Fig. 1. Alternatively, we can write a point $x \in \mathbb{R}$ as in decimal form

$$x = a_0.a_1a_2... , \tag{7}$$

where $a_0, a_1, a_2...$ are integers. The map is now $f(x) = 0.a_2...$ and so x is a fixed point if

$$a_0.a_1a_2... = 0.a_2a_3... , \tag{8}$$

This equation gives $a_0 = 0$, $a_1 = a_2$, $a_2 = a_3$ etc. The only numbers satisfying this are on the form $x^* = 0$, $x^* = 0.11111... = \frac{1}{9}$, $x^* = 0.2222... = \frac{2}{9}$,..., and $x^* = 0.99999... = 1$. Thus we have ten fixed points.

c) Let x_p be a rational number consisting of zeros except it contains a one on the first position after the period, on the $(p + 1)$th position after the period, the $(2p + 1)$ position after the period and so forth:

$$x_p = 0.100...100...100... \tag{9}$$

where the ellipsis indicate the remaining $p - 3$ zeros. It is clear that x_p is mapped onto x_p after exactly p iterations of the map. Hence x_p provides us with a period-p cycle. Since $f'(x) = 10$ for the decimal shift map, the orbits are unstable.

d) Since irrational numbers $\mathbb{R}\backslash\mathbb{Q}$ are aperiodic, they provide us with aperiodic orbits. Since there are infinitely many irrational numbers, we have infinitely many aperiodic orbits.

Problem 10.4.3

The map is given by

$$x_{n+1} \quad = \quad 1 - r x_n^2 \; . \tag{10}$$

A superstable 3-cycle (p, q, s) has by definition $\frac{d}{dx} f[f[f(x)]] = 0$. Using the chain rule, this is equivalent to $f'(x) = 0$, where $x = p, q$, or s. In the present case, we have $f(x) = 1 - rx^2$ and so $f'(x) = 0$ yields $-2rx = 0$, i.e. $x = 0$. Moreover

$$f[f[f(x)]] \quad = \quad 1 - r[1 - r(1 - rx^2)^2]^2 \; . \tag{11}$$

The 3-cycle satisfies $f[f[f(x)]] = x$, where $x = 0$ is in the cycle. Inserting $x = 0$ into $f^3(x) = x$ gives

$$\underline{\underline{1 - (1 - r)^2 r = 0}} \; , \tag{12}$$

which is the sought polynomial.