TFY4305 solutions exercise set 18 2014

Exam 2012 problem 3

a) Clearly $x = 0$ is a fixed point of the tent map $t(x)$. The stability is given by $|t'(0)|$. Since $t'(0) = r$, we find that $x = 0$ is stable for $\underline{\underline{r < 1}}$. For $r = 1$, $x = 0$ is marginally stable. Moreover, all points $x \leq \frac{1}{2}$ are fixed points and $x = 0$ is thereforeLiapunov stable. Using a cobweb, one can show that $x_n \to 0$ for all $x \in [0, 1]$ when $r < 1$. Thus $x = 0$ is globally stable for $\underline{\underline{r < 1}}$.

b) Using the definition of the tent map, one finds $q = t(p)$ $p = t(q)$ if $0 \leq p \leq \frac{1}{2}$ and $\frac{1}{2} \leq q \leq 1$. The inequality $q \geq \frac{1}{2}$ yields $r^2 \geq 1$, i.e. $r \geq 1$. For $r = 1$, we have $p = q$ such that the period-2 cycle is born at $r = 1$ (when the fixed point becomes marginally stable). The values for which it exists is therefore $\underline{\underline{r \geq 1}}$.

c) The stability of the period-2 cycle is given by $|\frac{d}{dx}t(t(x))|_{x=p}| = |t'(p)t'(q)|$. Since $t'(x) = r$ for all values of x, we $|t'(p)t'(q)| = r^2 > 1$ and the 2-cycle is is always *unstable* The Liapunov exponent is easy to calculate

$$\begin{aligned} \lambda &= \lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \ln|f'(x_i)| \\ &= \ln r \ . \end{aligned} \tag{1}$$

The tent map exhibits chaos if $\lambda > 0$, i.e. for $\underline{\underline{r > 1}}$.

Problem 10.3.6

The map is given by

$$x_{n+1} = rx_n - x_n^3 \ . \tag{2}$$

a) The fixed points are given by $x = rx - x^3$ and we notice that the origin is always a fixed point. The other possible fixed points satisfy

$$x^2 - (r-1) \;=\; 0\,. \tag{3}$$

or $\underline{\underline{x = \pm\sqrt{r-1}}}$. These fixed point exist for $r \geq 1$. The stability is given by

$$f'(x) \;=\; r - 3x^2\,, \tag{4}$$

$f'(0) = r$ and so the origin is stable for $|r| < 1$. Similarly $f'(\pm\sqrt{r-1}) = 3 - 2r$. These fixed points are stable for $\underline{\underline{1 < r < 2}}$.

b) The points x of two-cycles are roots of the polynomial

$$f[f(x)] \;=\; x\,, \tag{5}$$

or

$$r(rx - x^3) - (rx - x^3)^3 \;=\; x\,. \tag{6}$$

Since $f(x) - x$ is a factor in this polynomial, we can rewrite Eq. (6) by long division. This yields

$$x(x^2 - r + 1)(x^2 - r - 1)(x^4 - rx^2 + 1) \;=\; 0\,. \tag{7}$$

The first two factors give the fixed points of $f(x)$ and so the 2-cycles are found by the zeros of the third and fourth term. The third term yields

$$x_\pm \;=\; \underline{\underline{\pm\sqrt{r+1}}}\,, \tag{8}$$

and these exist for $r \geq -1$. The fourth term yields $\bar{x}^2 = \frac{r\pm\sqrt{r^2-4}}{2}$ or

$$\bar{x}_\pm \;=\; \underline{\underline{\pm\left[\frac{r \pm \sqrt{r^2-4}}{2}\right]^{\frac{1}{2}}}}\,. \tag{9}$$

These solutions exist for $r \geq 2$.

c) The derivative of $f[f(x)]$ evaluated at the fixed points $x_\pm$ reduces to

$$\begin{aligned} f'(f'(x_\pm)) &= f'(x_-)f'(x_+) \\ &= (3 + 2r)^2\,. \end{aligned} \tag{10}$$

In the region $r \geq -1$, this is always larger than unity. Hence the 2-cycle is always unstable. The derivative of $f[f(x)]$ evaluated at the fixed points $\bar{x}_\pm$ reduces to

$$\begin{aligned} f'(f'(x_\pm)) &= f'(\bar{x}_-)f'(\bar{x}_+) \\ &= 9 - 2r^2\,. \end{aligned} \tag{11}$$

Hence the 2-cycle is stable for $2 < r < \sqrt{5}$. In particular it is superstable for $r = 3/\sqrt{2}$.

d) The bifurcation diagram is shown in Fig. 1.

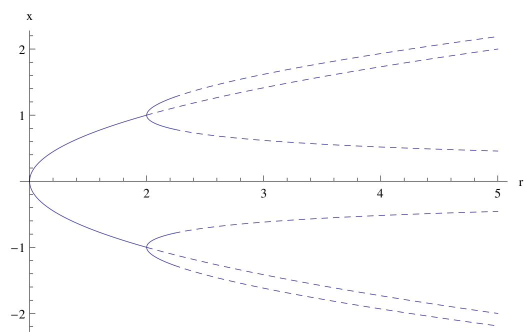

Figure 1: Partial bifurcation diagram.