# TFY4305 solutions exercise set 2 2014

## Problem 2.4.8

Gompertz' equation for tumor growth reads

$$\dot{N} = -aN\ln(bN) , \quad (1)$$

where $a, b >$ are parameters. The fixed points $N^*$ are given by

$$\begin{aligned} f(N) &= -aN\ln(bN) \\ &= 0 . \end{aligned} \quad (2)$$

This yields $N^* = 0$ and $N^* = 1/b$. The stability of the fixed point is given by the sign of $f'(N) = -a\ln(bN) - a$. This yields

$$f'(0) = \infty , \quad (3)$$

$$f'(1/b) = -a . \quad (4)$$

Thus the origin is unstable and $N^* = 1/b$ is stable.

**Comments**: The exact solution is

$$N(t) = \underline{\underline{\frac{1}{b}e^{\ln(N_0 b)e^{-at}}}} . \quad (5)$$

The solution satisfies $N(0) = N_0$ and

$$\lim_{t\to\infty} = \frac{1}{b} . \quad (6)$$

Fig. 1 shows the data points for tumor growth in a laboratory experiment at NTNU. The parameters $a$ and $b$ have been fitted to the data points. The agreement is very good.

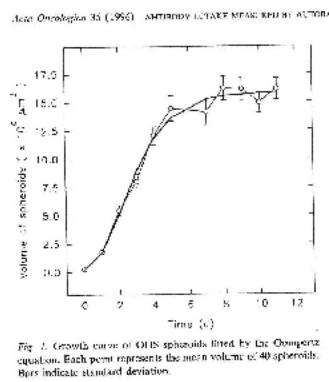

Figure 1: Tumor growth.

## Problem 2.5.1

a) The dynamics is governed by

$$\dot{x} = -x^c . \tag{7}$$

The origin is a fixed point only for $c > 0$. The stability is given by

$$f'(x) = -cx^{c-1} . \tag{8}$$

This implies that $f'(0) = -\infty$ for $0 < c < 1$. The flow is always towards the origin since $f'(x) < 0$ for $x > 0$ and so $x = 0$ is stable. For $c = 1$, $f'(0) = -1$ and for $c > 1$, $f'(0) = 0$. In the latter case $f'(x) < 0$ for $x > 0$ and the flow is towards the origin. Thus the origin is stable for all $c > 0$.

b) We can solve the differential equation exactly by separatation of variables. This yields

$$\int \frac{dx}{x^c} = -\int dt \, . \tag{9}$$

Integration yields

$$\frac{x^{1-c}}{1-c} = -t + K \, , \qquad c \neq 1 \, , \tag{10}$$

where $K$ is an integration constant. Using the initial condition $x(0) = x_0$, we can determine $K$ and find

$$x(t) = \underline{\underline{\left[(c-1)t + x_0^{1-c}\right]^{\frac{1}{1-c}}}} \, . \tag{11}$$

We must distinguish between two cases:

i) $c > 1$:
In this case the exponent $1/(1-c) < 0$ and this tells us that it takes infinitely long to reach the origin.

ii) $0 < c < 1$:
In this case the exponent $1/(1-c) > 0$ and this tells us that it takes us a finite amount of time $t^*$ to reach the origin. The equation for $t^*$ is $x(t^*) = 0$ or

$$(1-c)t^* = x_0^{1-c} \, . \tag{12}$$

This yields

$$t^* = \underline{\underline{\frac{x_0^{1-c}}{1-c}}} \, . \tag{13}$$

For $x_0 = 1$, we find

$$t^* = \underline{\underline{\frac{1}{1-c}}} \, . \tag{14}$$

Finally, for $c = 1$, the solution is

$$x(t) = x_0 e^{-t} \, , \tag{15}$$

and so it takes infinitely long time to reach the origin.

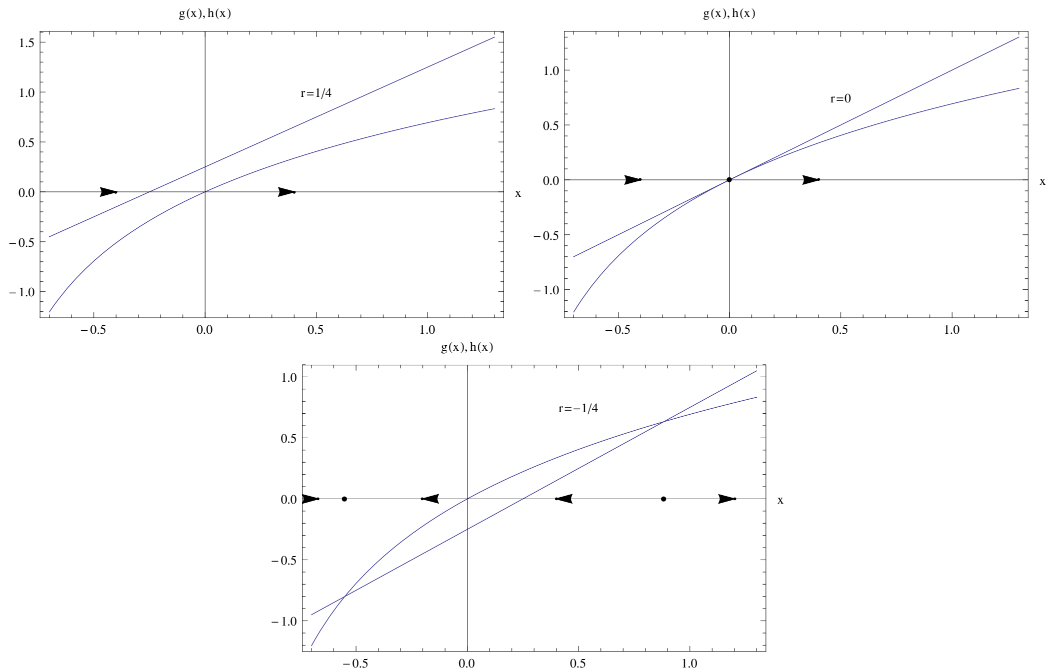


Figure 2: The function $g(x)$ for $r = 1/4$, $r = 0$, and $r = -1/4$. The number of fixed points depends on the parameter $r$. $r_c = 0$ is a bifurcation point.

# Problem 3.1.3

The equation is

$$\dot{x} \quad = \quad r + x - \ln(1 + x) \ . \tag{16}$$

In Fig. 2, we have plotted the function $g(x) = r + x$ for three different values of $r$ as well as the function $h(x) = \ln(1 + x)$.

We note that $g(x)$ crosses the $y$-axis at $r$ and so there is one fix point for $r = 0$. For $r > 0$, there are no fixed points and for $r < 0$ there are two fixed points. Hence $r = 0$ is a bifurcation point. One of the fixed points $x_1^*$ lies in the interval $(-1, 0]$ and the other $x_2^*$ in the interval $[0, \infty]$. Since $g(x) > h(x)$ for $x < x_1^*$ and $g(x) < h(x)$ for $x < x_1^*$ and $x_1^* < x < x_2^*$, $x_1^*$ is a stable fixed point. Since $g(x) < h(x)$ for $x_1^* < x < x_2^*$ and $g(x) > h(x)$ for $x > x_2^*$, $x_2^*$ is an unstable fixed point.

Finally, expanding the function around $x = 0$, we obtain

$$\begin{aligned} \dot{x} \quad &\approx \quad r + x - \left(x - \frac{1}{2}x^2\right) \\ &= \quad r + \frac{1}{2}x^2 \ . \end{aligned} \tag{17}$$

After rescaling of $x$, this is the same function as in Example 3.1 in the textbook. Thus a saddle-point bifurcation takes place at $r = 0$.

The bifurcation diagram is shown in Fig. 3.

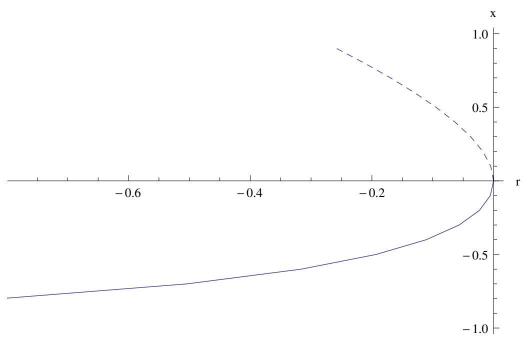


Figure 3: Bifurcation diagram.

# Problem 3.2.2

In Fig. 4, we plot the function $g(x) = rx$ for three different values of $r$ as well as the function $h(x) = \ln(1 + x)$ .

It is clear that $x = 0$ is a fixed point for all values of $r$. For $r < 1$ there is a second fixed point $x_2^* > 0$ and for $r > 1$ there is a second fixed point $x_1^* < 0$. Since $f'(x) = r - 1$, it follows that the origin is stable for $r < 1$ and unstable for $r > 1$. For $r = 1$, $g(x) > h(x)$ for all nonzero $x$ and so $x = 0$ is half stable. Moreover, for $r < 1$, the fixed point $x_2^*$ is unstable since $g(x) > h(x)$ for $x > x_2^*$ and $g(x) < h(x)$ for $0 < x < x_2^*$. Similar arguments show that $x_1^*$ is a stable fixed point for $r > 1$. Finally, expanding the function $f(x)$ around the origin yields

$$\begin{aligned} f(x) &\approx rx - \left(x - \frac{1}{2}x^2\right) \\ &= (r-1)x + \frac{1}{2}x^2 \ . \end{aligned} \tag{18}$$

After rescaling this is of the same form as Eq. (1) in Sec. 3.2 in the textbook and shows that $r = 1$ is a transcritical bifurcation. The bifurcation diagram is shown in Fig. 5.

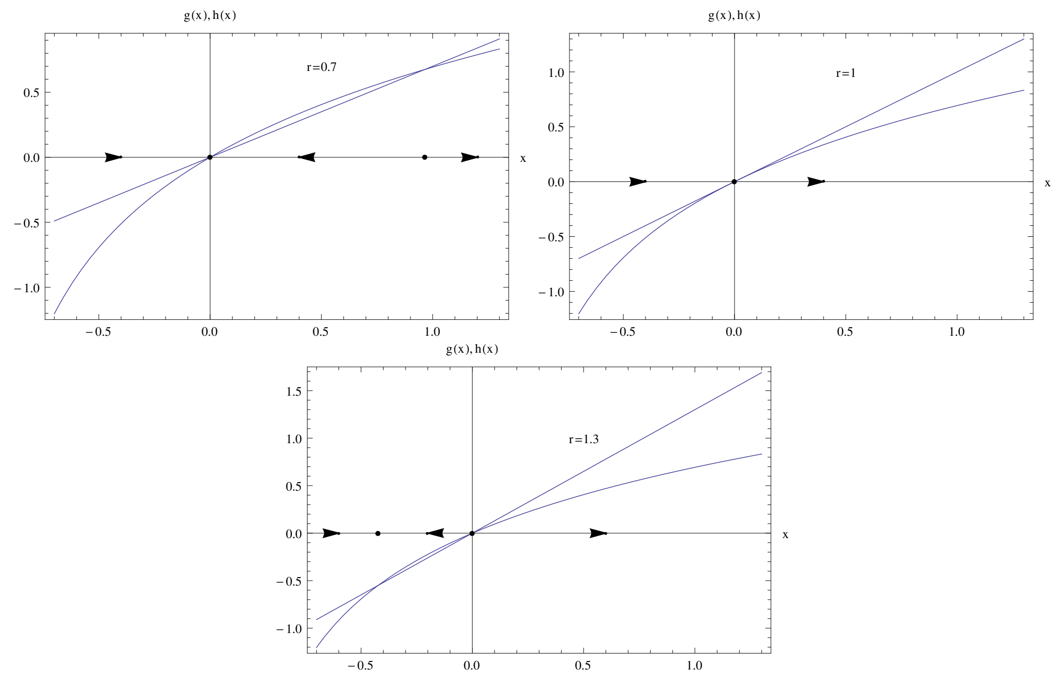


Figure 4: The function $g(x)$ for $r = 0.7$, $r = 1$, and $r = 1.3$. Transcritical bifurcation for $r_c = 1$.

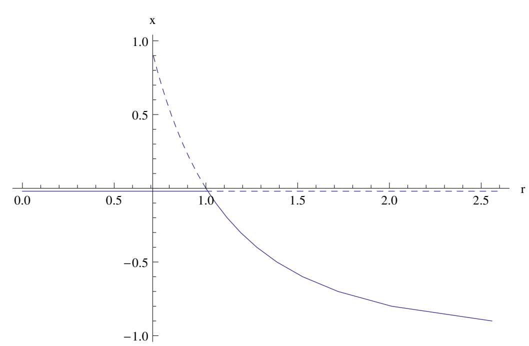


Figure 5: Bifurcation diagram.