TFY4305 solutions exercise set 20 2014

Problem 11.2.4

a) The set of rational numbers has zero measure (see 12.2.2). We first make a list of the rationals, $Q = c_1, c_2, ...$ and cover the first element by an interval of length ϵ, the second by an interval of length ϵ^2 and so forth. The total length of the intervals are

$$\begin{aligned} L &= \sum_{n=1}^{\infty} \epsilon^n \\ &= \frac{1}{1-\epsilon} - 1 \\ &= \frac{\epsilon}{1-\epsilon} \, , \end{aligned} \tag{1}$$

which can be made arbitrarily small. The measure of Q is therefore zero. The remainder, i. e. the irrational numbers must therefore have measure one.

b) We know that the reals are uncountable. Assume now that the irrational numbers are countable, i. e. we can make a list: $\mathbb{R}\backslash\mathbb{Q} = d_1, d_2, ...$ Then the reals can be written $R = c_1, d_1, c_2, d_2, ...$ and is therefore countable [1] This is a contradiction and the irrational numbers must therefore be uncontable.

c) Yes. Totally disconnected means that $\mathbb{R}\backslash\mathbb{Q}$ does not contain any intervals. Assume it does. There is therefore an interval around a point $x_0 \in \mathbb{R}\backslash\mathbb{Q}$,$(x_0 - \epsilon, x_0 + \epsilon)$ such that $(x_0 - \epsilon, x_0 + \epsilon) \subset \mathbb{R}\backslash\mathbb{Q}$. However, we know that the rational numbers are dense in R and so in any interval around any x_0 there are rational numbers and they do not belong to $\mathbb{R}\backslash\mathbb{Q}$. In other words, there is no interval around x_0 such that $(x_0 - \epsilon, x_0 + \epsilon) \subset \mathbb{R}\backslash\mathbb{Q}$.

d) No. If it would contain an isolated point x_0, there must exist an ϵ such that $(x_0 - \epsilon, x_0 + \epsilon)$

[1] The union of two countable sets must be countable. We know there is mapping $f(n) = c_n$ and $g(n) = d_n$. Thus we define $h(n) = f(n)$ if n is even and $h(n) = g(n)$ if n is odd. That gives the list above and the contradiction.

is a not subset of $\mathbb{R}\backslash\mathbb{Q}$. On the other hand, we know that we can find an irrational number arbitrarily close to any number in $[0, 1]$ [2]. But this means that we can always find a point $y_0 \in (x_0 - \epsilon, x_0 + \epsilon)$ and y_0 being irrational. This shows that there are no isolated points.

Problem 11.3.4

Divide your segment into 10 equally long pieces. All numbers in $[0, 1]$ are assigned one piece according to the first digit. Remove every other segment starting with the leftmost one. Then you have removed all numbers with the first digit being odd. Repeat ad infinitum. If we scale the original segment by a factor of 10, we need five segments to cover the new set. Thus the fractal dimension becomes

$$d = \underline{\underline{\frac{\ln 5}{\ln 10}}} \,. \tag{2}$$

[2]The irrational numbers are even "denser" in $\mathbb{R}$ than $\mathbb{Q}$.