TFY4305 solutions exercise set 6 2014

Problem 5.1.9

a) The dynamics of the system is governed by the equations

$$\dot{x} = -y \,, \tag{1}$$
$$\dot{y} = -x \,. \tag{2}$$

The velocity field $v = (-y, -x)$ is shown in Fig. 1.

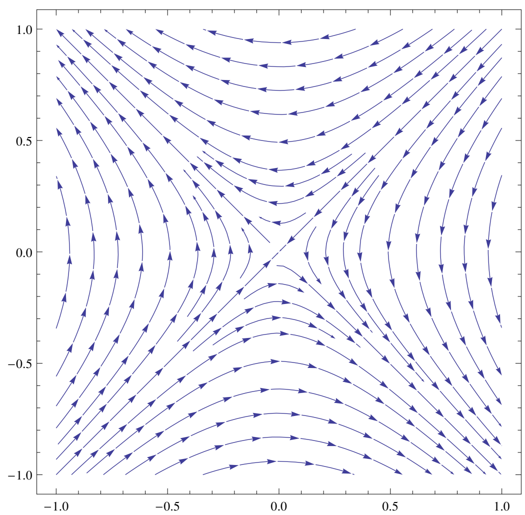

Figure 1: Velocity field for problem 5.1.9.

b) We have

$$x\dot{x} - y\dot{y} = -xy + xy = 0. \tag{3}$$

This implies that

$$\frac{d}{dt}(x^2 - y^2) = 0 \tag{4}$$

or after integration $x^2 - y^2 = C$.

c) The matrix is

$$A = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} . \tag{5}$$

we have to find the eigenvalues and eigendirections of A.

The eigenvalues satisfy $\lambda^2 - 1 = 0$, i. e. $\lambda = \pm 1$. For $\lambda = 1$, the eigenvector is (1,-1) and For $\lambda = -1$, the eigenvector is (1,1). Since $\Delta = -1$, the origin is a saddle point. The solutions are

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \underline{\underline{\begin{pmatrix} 1 \\ -1 \end{pmatrix} e^t}} , \tag{6}$$

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \underline{\underline{\begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-t}}} . \tag{7}$$

Thus the line $x = y$ is stable manifold and the line $x = -y$ is the unstable manifold.

d) We define $u = x + y$ and $v = x - y$. Note that up to a scaling of two, this is a rotation by a rotation matrix with angle $\theta = \frac{1}{2}\pi$. Taking the derivative of these equations and using the expressions for $\dot{x}$ and $\dot{y}$, we find

$$\dot{u} = -u , \tag{8}$$

$$\dot{v} = v . \tag{9}$$

The solutions are $\underline{\underline{u = u_0 e^{-t}}}$ and $\underline{\underline{v = v_0 e^t}}$, where u_0 and v_0 are initial conditions.

e) The equation for the stable manifold is seen to be $v = 0$, i.e. $x = y$. Similarly, the equation for the unstable manifold is seen to be $u = 0$, i.e. $x = -y$. This is in agreement with c).

f) Inverting the relations between the old and new coordinates, we obtain $x = \frac{1}{2}(u + v)$ and $y = \frac{1}{2}(u - v)$. This yields

$$x(t) = \frac{1}{2}(u_0 e^{-t} + v_0 e^t) , \tag{10}$$

$$y(t) = \frac{1}{2}(u_0 e^{-t} - v_0 e^t) . \tag{11}$$

Using $u_0 = x_0 + y_0$ and $v_0 = x_0 - y_0$, we can write

$$x(t) = \underline{\underline{x_0 \cosh t - y_0 \sinh t}} , \tag{12}$$

$$y(t) = \underline{\underline{y_0 \cosh t - x_0 \sinh t}} . \tag{13}$$

Problem 5.1.10

b) The equations are

$$\dot{x} = 2y \, , \tag{14}$$
$$\dot{y} = x \, . \tag{15}$$

If we start at $t = 0$ somewhere in the first quadrant, we have $x_0 > 0$ and $y_0 > 0$. In the first quadrant $\dot{x} > 0$ and $\dot{y} > 0$, which means that a tracjectory moves up and to the right forever, no matter how close (x_0, y_0) is to the origin. It therefore cannot be (Liapunov)stable or attracting.

c) The equations are

$$\dot{x} = 0 \, , \tag{16}$$
$$\dot{y} = x \, . \tag{17}$$

Integration of the first equation gives $x = C_1$ and so $\dot{y} = C_1$. Integration gives $y = C_1 t + C_2$ and the initial point is $(x_0, y_0) = (C_1, C_2)$. If $C_1 \neq 0$, we see the solution wanders off to infinity. So if you start off the x-axis and arbitrarily close to the origin, the solution wanders off to infinity and so the system cannot be stable nor attracting.

d) The equations are

$$\dot{x} = 0 \, , \tag{18}$$
$$\dot{y} = -y \, . \tag{19}$$

The solution is $x = C_1$ and $y = y_0 e^{-t}$. As $t \to \infty$, we end up at $(C_1, 0)$. The origin is therefore not attracting ($x_0 = C_1$ can be arbitrarily close to $x = 0$ and we will still not end up at $(0, 0)$). However, the origin is Liapunov stable since $x^2 + y^2 \leq x_0^2 + y_0^2$ and we can choose $\delta = \epsilon$.