TFY4305 solutions exercise set 7 2014

Problem 5.2.12

a) The equation that governs the dynamics is

$$L\ddot{I} + R\dot{I} + \frac{I}{C} = 0 \,. \tag{1}$$

We introduce the variables $x = I$ and $y = \dot{I} = \dot{x}$. The equation can then be written as

$$\dot{x} = y \,, \tag{2}$$

$$\dot{y} = -\frac{R}{L}y - \frac{1}{LC}x \,. \tag{3}$$

b) The matrix is

$$A = \begin{pmatrix} 0 & 1 \\ -\frac{1}{LC} & -\frac{R}{L} \end{pmatrix} . \tag{4}$$

The eigenvalues of A satisfy the equation

$$\lambda^2 + \frac{R}{L}\lambda + \frac{1}{LC} = 0 \,. \tag{5}$$

The solutions are

$$\lambda_{1,2} = \underline{\underline{-\frac{R}{2L} \pm \frac{R}{2L}\sqrt{1 - 4\frac{L}{R^2C}}}} \,. \tag{6}$$

Both eigenvalues have a negative real part for $R > 0$ irrespective of the values of L, C as long as they are positive. Both eigensolutions decay monotonically to $(x, y) = (0, 0)$. The origin is therefore <u>asymptotically stable</u>. For $R = 0$, the solutions reduce to

$$\lambda_{1,2} = \pm i\sqrt{\frac{1}{LC}} \,, \tag{7}$$

and are purely imaginary. This implies a harmonic-oscillator like solution (ellipse) and the origin is <u>neutrally stable</u>. Since $R = 0$ the total energy (stored between the plates and in the magnetic field of the coil) of the system is conserved.

c) If $R^2C-4L>0$, we have two real and negative eigenvalues. The fixed point is therefore a stable node. The current dies out without any oscillations (overdamped case) If $R^2C-4L<0$ we have two complex eigenvalues whose real part is negative. The fixed point is therefore a stable spiral. The current dies out while oscillating (underdamped) If $R^2C=4L$, we have one real and negative eigenvalue. The only eigenvector is

$$v \;=\; \begin{pmatrix} 1 \\ -\frac{R}{2L} \end{pmatrix} . \tag{8}$$

The fixed point is a degenerate node. The current dies out in the shortest time possible without oscillations (critically damped). The phase portraits are shown in Fig. 1.

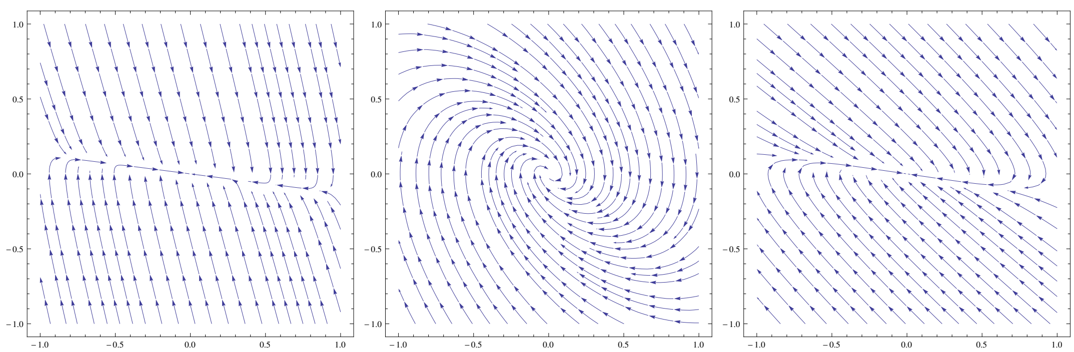

Figure 1: Phase portrait of problem 5.2.12. Left panel $R^2C-4L>0$, middle panel $R^2C-4L<0$, and right panel $R^2C-4L=0$.

Problem 5.2.13

a) The equation of motion is

$$m\ddot{x}+b\dot{x}+kx \;=\; 0 . \tag{9}$$

Defining $y=\dot{x}$, we obtain

$$\begin{aligned} \dot{y} &= \ddot{x} \\ &= -\frac{b}{m}y-\frac{k}{m}x , \end{aligned} \tag{10}$$

or in matrix form

$$A \;=\; \begin{pmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{b}{m} \end{pmatrix} . \tag{11}$$

b) The eigenvalues satisfy the equation

$$\lambda^2 + \frac{b}{m}\lambda + \frac{k}{m} \;=\; 0\,, \tag{12}$$

whose solutions are

$$\lambda \;=\; \frac{-\frac{b}{m} \pm \sqrt{(\frac{b}{m})^2 - 4\frac{k}{m}}}{2} \tag{13}$$

We have $\Delta = \frac{k}{m} > 0$ and $\tau = -\frac{b}{m}$, and so we have three cases:

1) The fixed point is a stable node if $\tau^2 - 4\Delta = (\frac{b}{m})^2 - 4\frac{k}{m} > 0$ since we have two real and negative eigenvalues.

2) It is the borderline case (degenerate node) if $\tau^2 - 4\Delta = (\frac{b}{m})^2 - 4\frac{k}{m} = 0$ since we have one real and negative eigenvalue. The eigenvector is then

$$v \;=\; \begin{pmatrix} 1 \\ -\frac{b}{2m} \end{pmatrix}\,. \tag{14}$$

3) It is a stable spiral if $\tau^2 - 4\Delta = (\frac{b}{m})^2 - 4\frac{k}{m} < 0$ since we have two complex eigenvalues with negative real part. The phase portraits are shown in Fig. 2.

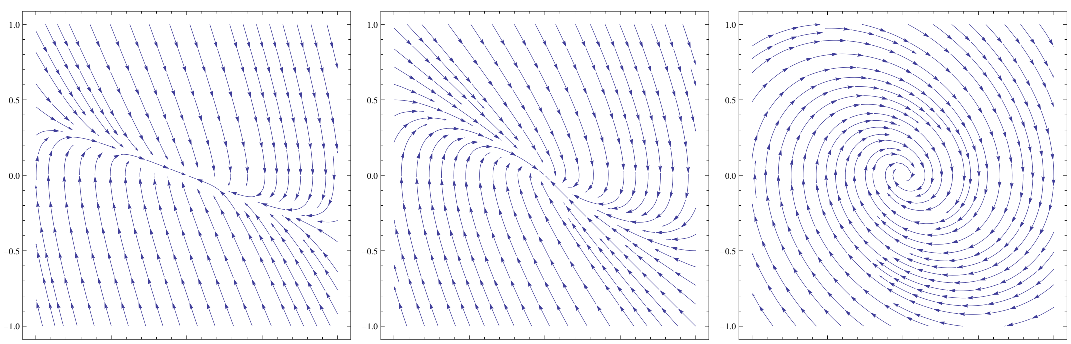

Figure 2: Phase portrait of problem 5.2.13. Left panel $(\frac{b}{m})^2 - 4\frac{k}{m} > 0$, middle panel $(\frac{b}{m})^2 - 4\frac{k}{m} = 0$, and right panel $(\frac{b}{m})^2 - 4\frac{k}{m} < 0$.

c) We have:

1) A node corresponds to the overdamped case - no oscillations.

2) The borderline case corresponds to the critically damped case - no oscillations. Comes to rest in the shortest time possible.

3) The spiral corresponds to the underdamped case - system oscillates with decreasing amplitude.