TFY4305 solutions exercise set 9 2014

Problem 6.5.6

The dynamics of an epidemic is given by

$$\dot{x} = -kxy \,, \tag{1}$$
$$\dot{y} = kxy - ly \,, \tag{2}$$

where $k, l > 0$ are constants. $x(t)$ is the size of the healthy population and $y(t)$ is the size of the sick population. The rate of change of the number of healthy people is proportional to how often sick and healthy people meet (xy, cf model for rabbits and sheep) proportional to k. The rate of change of the number is given by the same term minus the rate of people dying ly.

a) The first equation gives $x = 0$ or $y = 0$. If $x = 0$, the second equation gives $y = 0$, and if $y = 0$, the second equation is automatically satisfed. Thus there is a line of fixed points given by $\underline{\underline{y = 0}}$. The Jacobian matrix at a fixed point $(x, 0)$ is given by

$$A(x,0) = \begin{pmatrix} 0 & -kx \\ 0 & kx - l \end{pmatrix} . \tag{3}$$

Thusthe eigenvalues are $\lambda = 0$ and $\lambda = kx - l$, and so $\tau = kx - l$ and $\Delta = 0$. The fixed points are Liapunov stable for $x < l/k$ and unstable for $x > l/k$. The special point $(l/k, 0)$ is half-stable (See phase portrait below). The eigenvector is

$$v = \begin{pmatrix} 1 \\ -1 + \frac{k}{l} \end{pmatrix} . \tag{4}$$

b) The nullclines for $\dot{x}$ are given by the axes and the nullclines for $\dot{y}$ are given by $y = 0$ and $x = l/k$.

c) Dividing the second equation by the first, we obtain

$$\frac{dy}{dx} = \frac{l}{k}\frac{1}{x} - 1 \,. \tag{5}$$

Integrating yields

$$y = \underline{\underline{\frac{l}{k} \ln x - x + C}} \; , \tag{6}$$

where C is an integration constant.

d) The phase portrait is shown in Fig. 1 with the parameters $k = l = 1$. We note that $\dot{x} < 0$ for all $x, y > 0$. If we start to the left of the line $x = l/k$, we will remain there. If we start to the right of it, we will cross it eventually. Once we are to the left of this line, $\dot{y}$ as well and so we will end up somewhere on the x-axis as $t \to \infty$.

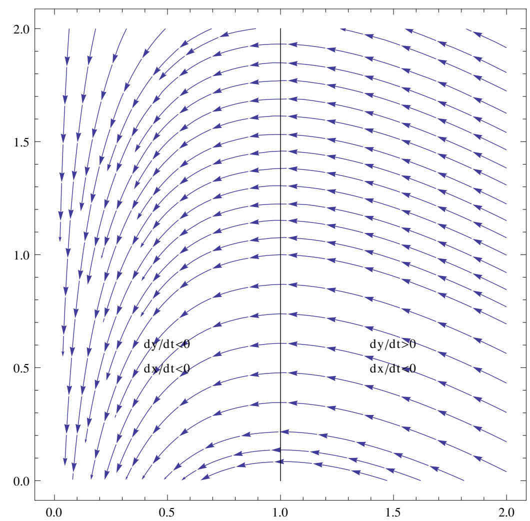

Figure 1: Phase portrait of problem 6.5.6 with $k = l = 1$. Vertical line is the nullcline $x = l/k$.

e) y increases if $\dot{y} > 0$. This implies that an epidemic occurs if $\dot{y}(0) > 0$. This requires $kx_0y_0 - ly_0 > 0$, i. e. if $\underline{\underline{x_0 > l/k}}$.

Problem 6.5.11

The dynamics is given by the equations

$$\dot{x} = y \; , \tag{7}$$

$$\dot{y} = -by + x - x^3 \; . \tag{8}$$

The fixed points are $(0, 0)$ and $(\pm 1, 0)$. The Jacobian matrix is given by

$$A(x, y) = \begin{pmatrix} 0 & 1 \\ 1 - 3x^2 & -b \end{pmatrix} \; . \tag{9}$$

Evaluated at the origin, we find

$$A(0,0) = \begin{pmatrix} 0 & 1 \\ 1 & -b \end{pmatrix}, \quad (10)$$

and so the eigenvalues are $\lambda = (-b \pm \sqrt{b^2+4})/2$. The origin is therefore a saddle. Evaluated at the fixed points $(\pm 1, 0)$, we find

$$A(\pm 1,0) = \begin{pmatrix} 0 & 1 \\ -2 & -b \end{pmatrix}, \quad (11)$$

and so $\lambda = (-b \pm \sqrt{b^2-8})/2$. Since $b \ll 1$, this shows that the eigevalues are complex and so the fixed point $(1,0)$ is a stable spiral. The phase portrait is shown in Fig. 2 with the parameters $b = 0.1$. We note that the homoclinic orbit that exists for $b = 0$ has disappeared due to damping.

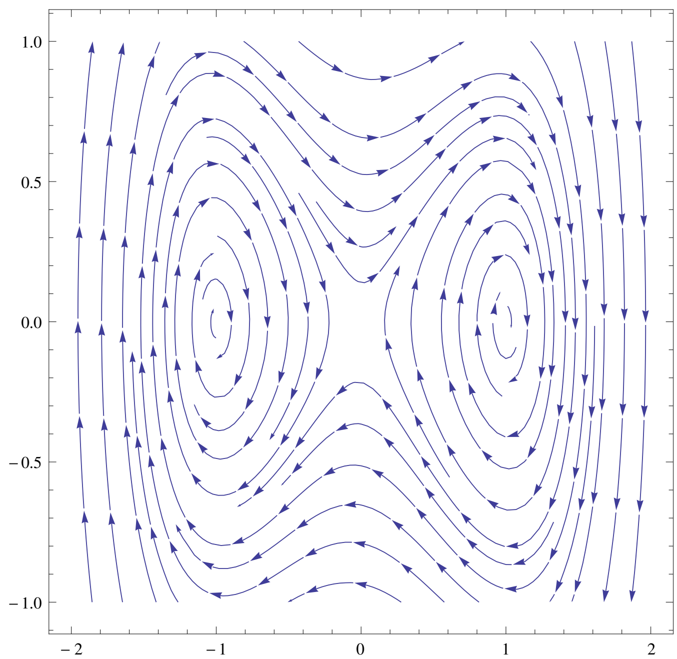

Figure 2: Phase portrait of problem 6.5.11 with $b = 0.05$.

In Fig. 3, we show the basin of attraction. The figure shows there are bands of initial conditions, where you end up at one of the fixed points. If you change the initial slightly, you may cross the boundary between two bands and end up at the other fixed point.

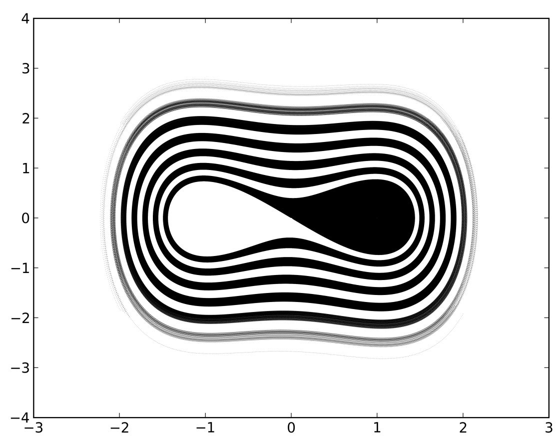

Figure 3: Basin of attraction.