of eight matrices

$$\mathbf{M}_1=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix},\qquad \mathbf{M}_2=\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix},\qquad \mathbf{M}_3=\begin{bmatrix}-1 & 0\\ 0 & -1\end{bmatrix},$$

$$\mathbf{M}_4=\begin{bmatrix}-1 & 0\\ 0 & 1\end{bmatrix},\qquad \mathbf{M}_5=\begin{bmatrix}0 & -1\\ 1 & 0\end{bmatrix},\qquad \mathbf{M}_6=\begin{bmatrix}0 & 1\\ -1 & 0\end{bmatrix},$$

$$\mathbf{M}_7=\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix},\qquad \mathbf{M}_8=\begin{bmatrix}0 & -1\\ -1 & 0\end{bmatrix}.$$

By explicit calculation it can be verified that the product of any two members of $\mathscr{G}$ is also contained in $\mathscr{G}$, so that axiom (a) is satisfied. Axiom (b) is automatically true for matrix multiplication, $\mathbf{M}_1$ is the identity of axiom (c) as it is a unit matrix, and finally axiom (d) is satisfied as

$$\mathbf{M}_1^{-1}=\mathbf{M}_1,\quad \mathbf{M}_2^{-1}=\mathbf{M}_2,\quad \mathbf{M}_3^{-1}=\mathbf{M}_3,\quad \mathbf{M}_4^{-1}=\mathbf{M}_4,$$
$$\mathbf{M}_5^{-1}=\mathbf{M}_6,\quad \mathbf{M}_6^{-1}=\mathbf{M}_5,\quad \mathbf{M}_7^{-1}=\mathbf{M}_7,\quad \mathbf{M}_8^{-1}=\mathbf{M}_8.$$

Example IV *The groups* U(N) *and* SU(N)

U(N) for $N \geqslant 1$ is defined to be the set of all $N \times N$ unitary matrices $\mathbf{u}$ with matrix multiplication as the group multiplication operation. SU(N) for $N \geqslant 2$ is defined to be the subset of such matrices $\mathbf{u}$ for which $\det \mathbf{u} = 1$, with the same group multiplication operation. (As noted in Appendix A, if $\mathbf{u}$ is unitary then $\det \mathbf{u} = \exp(i\alpha)$, where α is some real number. The "S" of SU(N) indicates that SU(N) is the "special" subset of U(N) for which this α is zero.)

It is easily established that these sets do form groups. Consider first the set U(N). As $(\mathbf{u}_1\mathbf{u}_2)^\dagger = \mathbf{u}_2^\dagger \mathbf{u}_1^\dagger$ and $(\mathbf{u}_1\mathbf{u}_2)^{-1} = \mathbf{u}_2^{-1}\mathbf{u}_1^{-1}$, if $\mathbf{u}_1$ and $\mathbf{u}_2$ are both unitary then so is $\mathbf{u}_1\mathbf{u}_2$. Again axiom (b) is automatically valid for matrix multiplication and, as the unit matrix $\mathbf{1}_N$ is a member of U(N), it provides the identity E of axiom (c). Finally, axiom (d) is satisfied, as if $\mathbf{u}$ is a member of U(N) then so is $\mathbf{u}^{-1}$.

For SU(N) the same considerations apply, but in addition if $\mathbf{u}_1$ and $\mathbf{u}_2$ both have determinant 1, Equation (A.4) shows that the same is true of $\mathbf{u}_1\mathbf{u}_2$. Moreover, $\mathbf{1}_N$ is a member of SU(N), so it is its identity, and $\mathbf{u}^{-1}$ is a member of SU(N) if that is the case for $\mathbf{u}$.

The set of groups SU(N) is particularly important in theoretical physics. SU(2) is intimately related to angular momentum and isotopic spin, as will be shown in Chapters 12 and 18, while SU(3) is now famous for its role in the classification of elementary particles

	$\mathbf{M}_1$	$\mathbf{M}_2$	$\mathbf{M}_3$	$\mathbf{M}_4$	$\mathbf{M}_5$	$\mathbf{M}_6$	$\mathbf{M}_7$	$\mathbf{M}_8$
$\mathbf{M}_1$	$\mathbf{M}_1$	$\mathbf{M}_2$	$\mathbf{M}_3$	$\mathbf{M}_4$	$\mathbf{M}_5$	$\mathbf{M}_6$	$\mathbf{M}_7$	$\mathbf{M}_8$
$\mathbf{M}_2$	$\mathbf{M}_2$	$\mathbf{M}_1$	$\mathbf{M}_4$	$\mathbf{M}_3$	$\mathbf{M}_8$	$\mathbf{M}_7$	$\mathbf{M}_6$	$\mathbf{M}_5$
$\mathbf{M}_3$	$\mathbf{M}_3$	$\mathbf{M}_4$	$\mathbf{M}_1$	$\mathbf{M}_2$	$\mathbf{M}_6$	$\mathbf{M}_5$	$\mathbf{M}_8$	$\mathbf{M}_7$
$\mathbf{M}_4$	$\mathbf{M}_4$	$\mathbf{M}_3$	$\mathbf{M}_2$	$\mathbf{M}_1$	$\mathbf{M}_7$	$\mathbf{M}_8$	$\mathbf{M}_5$	$\mathbf{M}_6$
$\mathbf{M}_5$	$\mathbf{M}_5$	$\mathbf{M}_7$	$\mathbf{M}_6$	$\mathbf{M}_8$	$\mathbf{M}_3$	$\mathbf{M}_1$	$\mathbf{M}_4$	$\mathbf{M}_2$
$\mathbf{M}_6$	$\mathbf{M}_6$	$\mathbf{M}_8$	$\mathbf{M}_5$	$\mathbf{M}_7$	$\mathbf{M}_1$	$\mathbf{M}_3$	$\mathbf{M}_2$	$\mathbf{M}_4$
$\mathbf{M}_7$	$\mathbf{M}_7$	$\mathbf{M}_5$	$\mathbf{M}_8$	$\mathbf{M}_6$	$\mathbf{M}_2$	$\mathbf{M}_4$	$\mathbf{M}_1$	$\mathbf{M}_3$
$\mathbf{M}_8$	$\mathbf{M}_8$	$\mathbf{M}_6$	$\mathbf{M}_7$	$\mathbf{M}_5$	$\mathbf{M}_4$	$\mathbf{M}_2$	$\mathbf{M}_3$	$\mathbf{M}_1$

Table 1.1 Multiplication table for the group of Example III.

tion relations between the basis elements of the corresponding *real Lie algebra*, as will be explained in detail in Chapter 10.

2 Groups of coordinate transformations

To proceed beyond an intuitive picture of the effect of symmetry operations, it is necessary to specify the operations in a precise algebraic form so that the results of successive operations can be easily deduced. Attention will be confined here to transformations in a real three-dimensional Euclidean space $\mathbb{R}^3$, as most applications in atomic, molecular and solid state physics involve only transformations of this type. (The generalization to Minkowski space–time will be introduced in Example V of Chapter 2, Section 7, and developed in more detail in Chapter 17.)

(a) *Rotations*

Let Ox, Oy, Oz be three mutually orthogonal Cartesian axes and let Ox', Oy', Oz' be another set of mutually orthogonal Cartesian axes with the same origin O that is obtained from the first set by a rotation T about a specified axis through O. Let (x, y, z) and (x', y', z')