TFY4215/FY1006

Delta function

Dirac's delta function can be loosely defined as

$$\delta(x) = \begin{cases} 0\,, & x \neq 0\,, \\ \infty\,, & x = 0\,. \end{cases} \tag{1}$$

in such a way that the area under the curve is constant

$$\int_{-\infty}^{\infty} \delta(x)\, dx \;=\; 1\,. \tag{2}$$

One can formally show that no such function exists. However, one can define a sequence of integrable functions $f_n(x)$, such that

$$\int_{-\infty}^{\infty} f_n(x)\, dx \;=\; 1\,, \tag{3}$$

for all n and such that $f_n(x)$ becomes increasingly concentrated around $x = 0$ as n increases. We can then define the integral of the δ-function by

$$\int_{-\infty}^{\infty} \delta(x)\, dx \;\equiv\; \lim_{n\to\infty} \int_{-\infty}^{\infty} f_n(x)\, dx\,. \tag{4}$$

A sequence of functions with these properties is called a δ-*sequence*. For example the sequence of Gaussians

$$f_n(x) \;=\; \frac{n}{\sqrt{\pi}} e^{-n^2x^2}\,, \tag{5}$$

has these properties. This is illustrated in Fig. 1.

Another example is the sequence of functions $g_n(x)$ defined by

$$g_n(x) \;=\; \begin{cases} \frac{n}{2}\,, & |x| < \frac{1}{n}\,, \\ 0\,, & |x| \geq \frac{1}{n}\,. \end{cases} \tag{6}$$

For proofs, the sequence $g_n(x)$ is easier to use. [1]

[1]Note that all identities and relations must be independent of the particular sequence one is using. This can be shown rigorously.

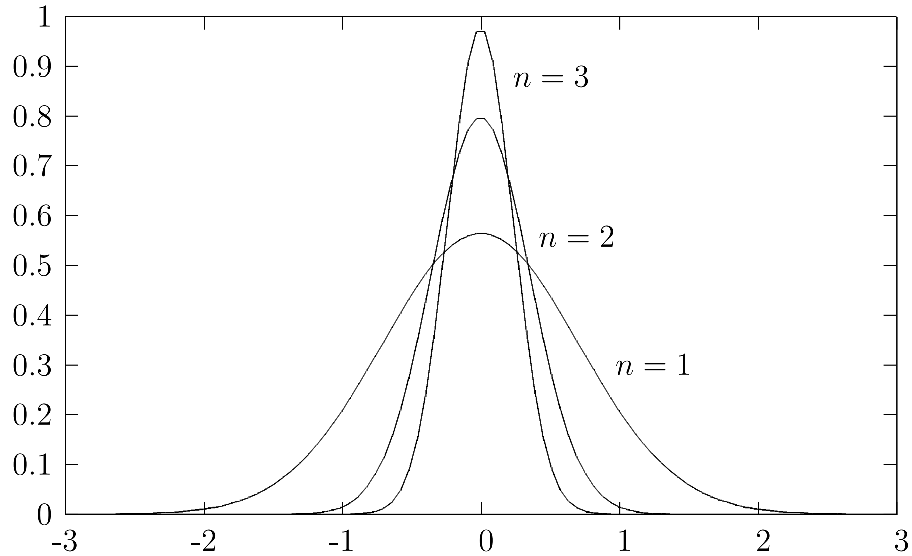

Figure 1: Sequence $f_n(x)$ of Gaussians which are increasingly concentrated around $x = 0$.

We are interested in calculating the integral

$$I \equiv \int_{-\infty}^{\infty} f(x)\delta(x)\, dx \, , \tag{7}$$

where $f(x)$ is some integrable function. Heuristically speaking, the δ-function picks up a contribution to the integral only when $x = 0$, and so our intuition tells us that the value of the integral should be $I = f(0)$. Let us confirm this expectation. In analogy with the definition Eq. (4), we define the integral I by

$$I \equiv \lim_{n\to\infty} \int_{-\infty}^{\infty} f(x) g_n(x)\, dx \tag{8}$$

Using the sequence (6), we obtain

$$\begin{aligned} \int_{-\infty}^{\infty} f(x) g_n(x)\, dx &= \frac{n}{2} \int_{-1/n}^{1/n} f(x)\, dx \\ &= f(\xi) \, , \end{aligned} \tag{9}$$

where $-\frac{1}{n} < \xi < \frac{1}{n}$. The last line follows from the mean-value theorem of calculus. In the limit $n \to \infty$, the value is $\xi = 0$ and so we obtain the important result

$$\boxed{\int_{-\infty}^{\infty} \delta(x) f(x)\, dx = f(0)} \tag{10}$$

Similarly, let us calculate the integral

$$I = \int_{-\infty}^{\infty} f(x)\delta(x-a)\, dx \ , \tag{11}$$

where a is a nonzero constant. Changing variable to $y = x - a$, we see that we must have $I = f(a)$:

$$\begin{aligned} I &= \int_{-\infty}^{\infty} f(y+a)\delta(y)\, dy \\ &= f(a) \ . \end{aligned} \tag{12}$$

We next consider the integral

$$I = \int_{-\infty}^{\infty} \delta(f(x))\, dx \ , \tag{13}$$

If the function $f(x)$ is nonzero for all $x \in (-\infty, \infty)$, the integral vanishes. Assume next that the function has a single zero at $x = x^*$. Moreover assume that $f'(x^*) > 0$ so that $f(x)$ is invertible in a small interval around x^*. Introducing the variable $y = f(x)$, the integral can be written as

$$\begin{aligned} I &= \int_{x^*-\Delta}^{x^*+\Delta} \delta(f(x))dx \\ I &= \int_{f(x^*)-f'(x^*)\Delta}^{f(x^*)+f'(x^*)\Delta} \delta(y)\frac{dx}{dy}dy \\ &= \int_{-f'(x^*)\Delta}^{f'(x^*)\Delta} \delta(y)[f^{-1}(y)]'dy \\ &= [f^{-1}(y)]'\Big|_{y=0} \\ &= \frac{1}{f'(x^*)} \ , \end{aligned} \tag{14}$$

where we in the last line have used $x = f^{-1}(y)$ and therefore $1 = [f^{-1}(y)]'f'(x)$. If $f'(x^*) < 0$, the integral changes sign, $I = -1/f'(x^*)$. The two cases can conveniently be summarized in the formula

$$I = \frac{1}{|f'(x^*)|} \ . \tag{15}$$

If the function $f(x)$ has more than one zero, one must simply sum over all the zeros x_i^* of $f(x)$ and we can write

$$\boxed{\delta(f(x)) = \sum_i \frac{1}{|f'(x_i^*)|}\delta(x - x_i^*)} \tag{16}$$

Example

Consider the following integral

$$I = \int_{-\infty}^{\infty} \delta(x^2 - a^2)\, dx \, . \tag{17}$$

We define $f(x) = x^2 - a^2$ and we then have $f'(x) = 2x$. The zeros of $f(x)$ are $x^* = \pm a$ and hence $|f'(x = x^*)| = 2|a|$. This implies that we can write

$$\begin{aligned} I &= \frac{1}{2|a|} \int_{-\infty}^{\infty} \left[\delta(x-a) + \delta(x+a)\right]\, dx \\ &= \frac{1}{|a|} \, . \end{aligned} \tag{18}$$

Note that the function $\delta(x^2)$ is too singular to be well defined. The problem is that the derivative of $f(x) = x^2$ vanishes at $x = 0$ which is the zero of the function itself and so Eq. (15) makes no sense. This also follows directly from Eq. (18).

Heaviside's step function

Let us define the function $\theta(x)$ by

$$\theta(x) = \int_{-\infty}^{x} \delta(x)\, dx \, . \tag{19}$$

Using the rules of delta-function calculus, we find [2]

$$\theta(x) = \begin{cases} 1\,, & x > 0\,, \\ 0\,, & x < 0\,. \end{cases} \tag{20}$$

which is nothing but Heaviside's step function, see Fig. 2.

From the fundamental theorem of calculus, one obtains

$$\theta'(x) = \delta(x)\, . \tag{21}$$

Eq. (21) is a formal result as the derivative of Eq. (20). $\theta'(x)$ is not defined for $x = 0$.

[2] The defining Eq. (19) may not hold for $x = 0$. It depends on the limiting procedure that defines the integral of $\delta(x)$. For example using one of the delta sequences from this chapter, it is clear that $\theta(0) = 1/2$, which is also a common definition.

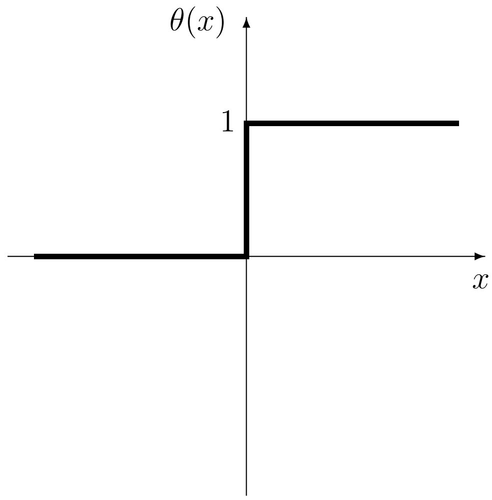

Figure 2: Heaviside step function.

Problems

1) Calculate the Fourier transform $f(p)$ of the delta function $\delta(x)$. Use this result to find an integral representation of $\delta(x)$.

2) Prove the *derivative sifting formula*:

$$\int_{-\infty}^{\infty} \delta'(x) f(x)\, dx \quad = \quad -f'(0)\ , \tag{22}$$

where $\delta'(x)$ is the derivative of the δ function and $f(x)$ is any differentiable function. Hint: use integration by parts and a *differentiable* δ-sequence, e.g. the Gaussian in Eq. (5).