NTNU NTNU
Faculty of Science and Technology
Department of Physics

Exam TFY4305 Nonlinear dynamics
Fall 2013

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Saturday December 21 2013
09.00h-13.00h

You may bring:
Approved calculator
Rottmann: Matematisk Formelsamling
Rottmann: Matematische Formelsammlung
Barnett & Cronin: Mathematical Formulae

The problem set is three pages. Read carefully. Ask if in doubt. Viel Glück!

Problem 1

Consider the mechanical system shown in Fig. 1. It consists of two masses m that can slide on a horizontal rod. Two massless ropes are connecting the mass M to the masses m. The distance between the masses m and the z-axis is R and the system rotates around the z-axis with angular frequency ω. In a) and b), the masses m slide frictionlessly on the rod. Conservation of angular momentum of the system can be written as a second-order equation for R,

$$\ddot{R} + \alpha g - \frac{\beta}{R^3} = 0 \,. \tag{1}$$

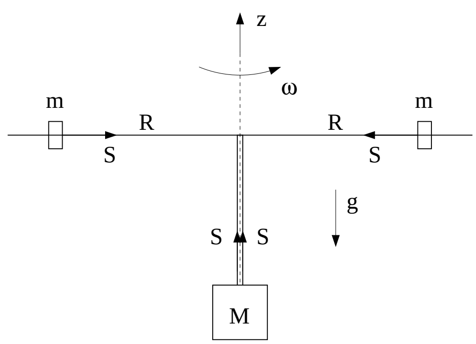

Figure 1: Set up in problem 1.

a) Show that Eq. (1) can be written as the system of two first-order equations given below:

$$\dot{R} = u \,, \tag{2}$$

$$\dot{u} = -\alpha g + \frac{\beta}{R^3} \,. \tag{3}$$

b) Find the fixed point (R^*, u^*) for this system. Linearize the system and classify the fixed point. Explain why linearization can be applied in this case.

c) In the remainder, there is friction between the masses m and the rod. Newton's second law of motion can then be written as

$$\ddot{R} + \gamma \dot{R} + \alpha g - \frac{\beta}{R^3} = 0 \,, \tag{4}$$

where $\gamma > 0$ is a parameter. Rewrite Eq. (4) as a system of first-order equations and find the fixed point. Classify the fixed point as a function of the parameter γ. Explain the trajectories in phase space close to the fixed point and give a physical interpretation.

Problem 2

Consider the set of equations

$$\dot{x} = a - x + x^2 y \,, \tag{5}$$
$$\dot{y} = b - x^2 y \,, \tag{6}$$

where $a > 0$ and $b > 0$ are parameters. We restrict ourselves to the region $x \geq 0$ og $y \geq 0$.

a) Find the fixed point for the system (5)–(6).

b) Find the equation for the curve that divides the a–b plane into a region where the fixed point is stable and another region where it is unstable. Sketch this curve in the a–b plane and indicate the two regions.

c) What type of bifurcation does the system go through as we cross the line you found in b)?

Problem 3

Consider the following model for the interaction between predators and prey:

$$\dot{x} = x[x(1-x) - y] \,, \tag{7}$$
$$\dot{y} = y(x - a) \,, \tag{8}$$

where $x(t)$ is the number of prey (sheep) and $y(t)$ is the number of predators (wolves), and $a \geq 0$ is a constant. Obviously, we must have $x(t) \geq 0$ and $y(t) \geq 0$.

a) Sketch the nullclines in the first quadrant. Use $a > 1$ in your sketch. Indicate the signs of $\dot{x}$ and $\dot{y}$ in the different regions.

b) Find the fixed points for the equations (7)–(8) and classify them.

c) Show that the wolves go extinct if $a > 1$. Hint: use the result in a).

d) A Hopf bifurcation occurs for a critical value a_c of a. Find a_c and find the frequency of the limit cycle for $a \sim a_c$.

Problem 4

Consider the map

$$x_{n+1} = 2x_n \pmod 1 . \tag{9}$$

a) Find the fixed points for the map (9).

b) Find the unique period-2 cycle. Is the cycle stable?

c) Find the points $x_0 \in [0, 1)$ such that

$$\lim_{n\to\infty} x_n = 0 . \tag{10}$$