TFY4305 solutions extra exercises 2014

Problem 2.3.3

Tumor growth can be modelled by Gompertz' law

$$\dot{N} = -aN\ln(bN) \,, \tag{1}$$

where $N(t)$ is proportional to the number of cells in the tumor, and $a, b > 0$ are parameters.

a) In order to interpret b we note that $N = 1/b$ is a stable fixed point since $f(1/b) = 0$ and $f(N) = -aN\ln(bN) > 0$ for $N < 1/b$ and $f(N) < 0$ for $N > 1/b$. Thus $1/b$ is the limiting size of the tumor. The parameter a is related to the proliferation ability (cell division) and depends on the availability of substrate, oxygen etc.

Interestingly the Gompertz equation can be solved exactly. Separation of variables gives

$$\frac{dN}{N\ln bN} = -adt \tag{2}$$

Integration yields

$$\ln[\ln(bN)] = -at + C \,, \tag{3}$$

where C is an integration constant. Exponentiating this expression twice gives

$$N(t) = \frac{1}{b}e^{e^{C}e^{-at}} \,. \tag{4}$$

Using the initial condition $N(0) = N_0$, one finds $N_0 = \frac{1}{b}e^{e^{C}}$ or $\ln(N_0 b) = e^{C}$. The solution can then be written as

$$N(t) = \underline{\underline{\frac{1}{b}e^{\ln(N_0 b)e^{-at}}}} \,. \tag{5}$$

The solution satisfies $N(0) = N_0$ and

$$\lim_{t\to\infty} = \frac{1}{b} \,. \tag{6}$$

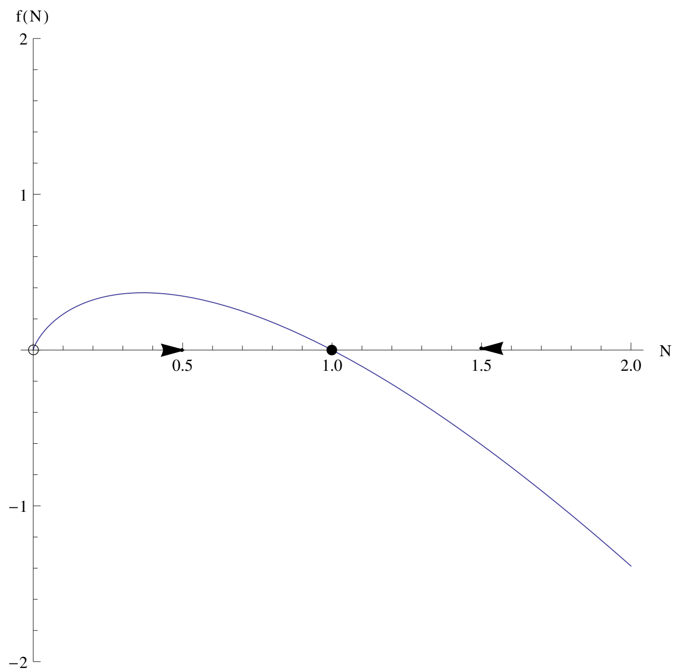

Figure 1: Vector field for the Gompertz model with $b = 1$.

We can also gain insight into the solution without knowing it. Taking the derivative of the Gompertz equation, we obtain

$$\begin{aligned} \ddot{N} &= -a\dot{N}(\ln(bN) + 1) \\ &= a^2 N \ln(bN)(\ln(bN) + 1)\ . \end{aligned} \tag{7}$$

Assume first that $N_0 > 1/b$. Then we know that $N(t)$ decreases monotonically to $N = 1/b$ as $t \to \infty$. In this case $\dot{N} < 0$ and so $\ddot{N} > 0$. This implies that there is no inflection point. Next assume that $N_0 < 1/b$. Then we that $N(t)$ increases monotonically to $N = 1/b$ as $t \to \infty$. In this case $\dot{N} < 0$ but $\ddot{N}$ changes sign at $N = \frac{1}{b}e^{-1}$. Thus there is an inflection point here.

b) The other fixed point of the differential equation is $N = 0$ and this fixed point is unstable. The vector field is shown in Fig. 1 for $b = 1$. The graph $N(t)$ is shown for $a = b = 1$ and two initial conditions $N_0 = 1/20 < 1/b$ and $N_0 = 1.2 > 1/b$ in Fig. 2. In the former case one can clearly see the inflection point as predicted above.

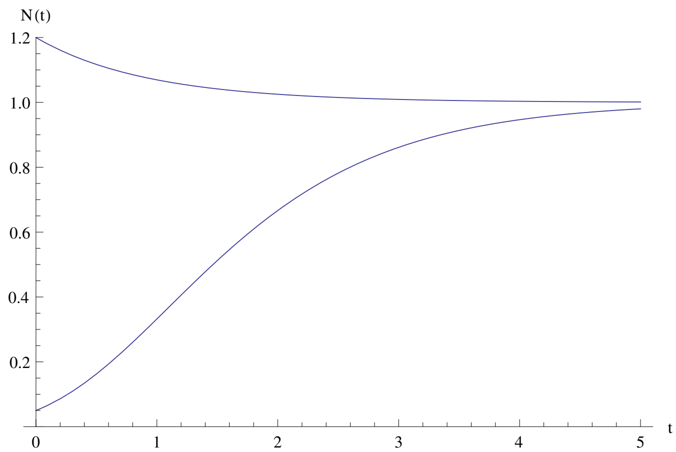

Figure 2: Solution $N(t)$ for $a = 1$ and two initial conditions $N_0 = 1/20 < 1/b$ and $N_0 = 1.2 > 1/b$.

Fig. 3 shows the data points for tumor growth in a laboratory experiment at NTNU. The parameters a and b have been fitted to the data points. The agreement is very good.

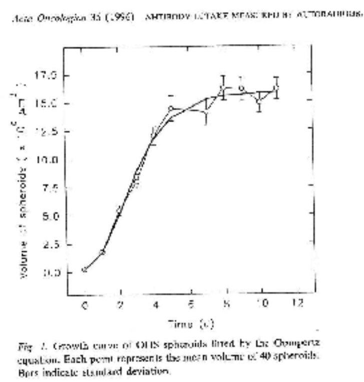

Figure 3: Tumor growth.

Problem 2.4.6

We have

$$\dot{x} = \ln x \,. \tag{8}$$

The fixed points are found by solving $f(x^*) = \ln x^* = 0$, which yields $\underline{\underline{x^* = 1}}$. Moreover

$$f'(x) = \frac{1}{x} \,, \tag{9}$$

and so $f'(1) = 1$. The fixed point $\underline{\underline{x^* = 1 \text{ is therefore unstable}}}$.

Problem 2.6.1

The point is that the harmonic oscillator is not a first-order system. It is a system of two coupled differential equations. Define $\dot{x} = y$. This yields

$$m\dot{y} = -kx \tag{10}$$
$$\dot{x} = y\ , \tag{11}$$

and we conclude that the system is two dimensional and so does not correspond to flow on the line.

Problem 2.7.5

The dynamics is governed by the equation

$$\dot{x} = -\sinh x\ . \tag{12}$$

The function $f(x) = -\sinh x$ is shown in Fig. 4. The only zero of $f(x)$, i. e. fixed point is

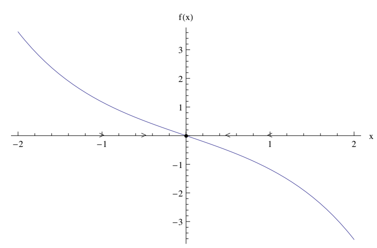

Figure 4: Function $f(x) = -\sinh x$.

$x = 0$. Since $f'(x) = -\cosh x$ and $f'(0) = -1$, the fixed point is stable. This can also be inferred from inspecting the graph.

The potential is given by integration of $dV/dx = \sinh x$, Integration yields

$$V(x) = \cosh x + C\ , \tag{13}$$

where C is an integration constant we set to zero.

The potential $V(x)$ is shown in Fig 5.

Problem 6.3.13

The dynamics is governed by the set of equations

$$\dot{x} = -y - x^3\ , \tag{14}$$
$$\dot{y} = x\ . \tag{15}$$

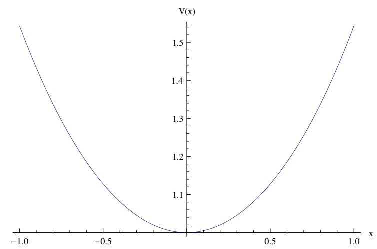

Figure 5: Function $V(x) = \cosh x$.

The Jacobian matrix is

$$A = \begin{pmatrix} -3x^2 & -1 \\ 1 & 0 \end{pmatrix} . \tag{16}$$

Evaluated at the origin, we find

$$A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} . \tag{17}$$

This yields the eigenvalues $\lambda_{1,2} = \pm i$, i. e. linearization predicts a center. In order to gain some insight into the flow, we switch to polar coordinates. One finds

$$\begin{aligned} \dot{r} &= \frac{x\dot{x} + y\dot{y}}{r} \\ &= -r^3 \cos^4 \theta \ , \end{aligned} \tag{18}$$

which shows that $\dot{r} \leq 0$ and r is a nonincreasing function. Only for $\theta = \pi/2$ and $\theta = 3\pi/2$ is the radial flow vanishing. Moreover, one finds

$$\begin{aligned} \dot{\theta} &= \frac{x\dot{y} - y\dot{x}}{r^2} \\ &= 1 + r^2 \cos^3 \theta \sin \theta \ . \end{aligned} \tag{19}$$

$\dot{\theta}$ is always nonzero when $\dot{r} = 0$ and so the origin is the only fixed point. For $r < 1$, $\dot{\theta} > 0$ and so the trajectory must be spiraling inwards. The phase portrait is shown in Fig. 6

Problem 6.3.14

The dynamics is governed by the set of equations

$$\dot{x} = ax^3 - y \ , \tag{20}$$

$$\dot{y} = x + ay^3 \ . \tag{21}$$

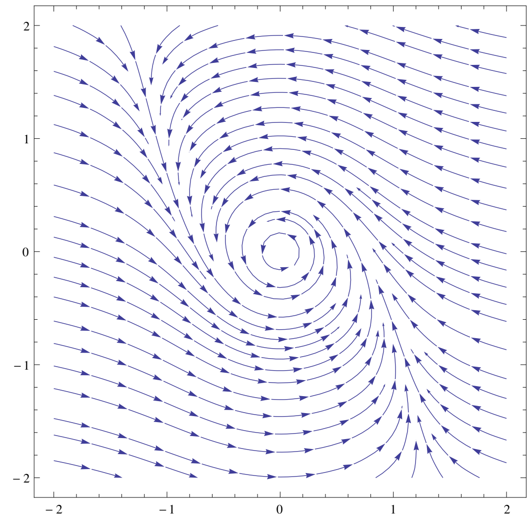

Figure 6: Phase portrait of problem 6.3.13.

a) The Jacobian matrix is

$$A = \begin{pmatrix} 3ax^2 & -1 \\ 1 & 3ay^2 \end{pmatrix} . \tag{22}$$

Evaluated at the origin, we obtai

$$A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} . \tag{23}$$

The eigenvalues are $\lambda_{1.2} = \pm i$ and so it is a linear center. However, note that this is a borderline case and so we have to be careful. Going to polar coordinates, we find

$$\dot{r}r = a(x^4 + y^4) ,$$

Thus for $r \neq 0$, $\dot{r} > 0$ for $a > 0$ and $\dot{r} < 0$ for $a < 0$. Hence r is increasing for $a > 0$ and decreasing for $a < 0$. Moreover

$$\dot{\theta} = 1 - ar^2 \cos\theta \sin\theta (\cos^2\theta - \sin^2\theta) , \tag{24}$$

and so θ is an increasing function when r is sufficiently small. Hence the origin is an unstable spiral for $a > 0$ and a stable spiral for $a < 0$. In both cases it spirals counterclockwise. Finally for $a = 0$, the origin is a center since the equation reduces to that of a harmonic oscillator (with appropriately scaled coordinates).

Problem 6.5.1

The second-order equation is

$$\ddot{x} = x^3 - x . \tag{25}$$

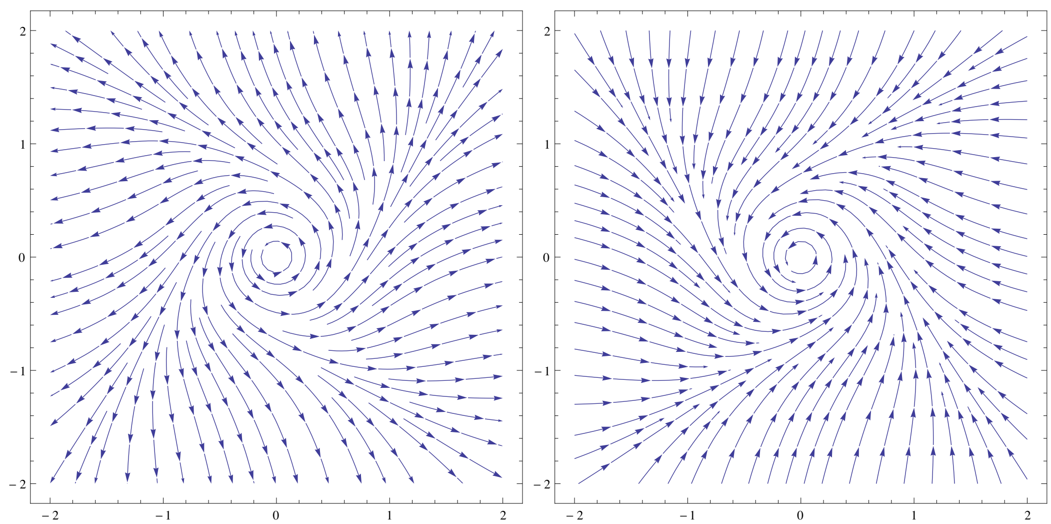

Figure 7: Phase portrait of problem 6.3.14. Left plot is for $a = 1$ and right plot is for $a = -1$.

This can be written as a set of first-order equations

$$\dot{x} = y \, , \tag{26}$$
$$\dot{y} = x^3 - x \, . \tag{27}$$

a) The fixed points are $(x, y) = \underline{\underline{(0, 0)}}$ and $(x, y) = \underline{\underline{(\pm 1, 0)}}$. The Jacobian matrix is

$$A = \begin{pmatrix} 0 & 1 \\ 3x^2 - 1 & 0 \end{pmatrix} . \tag{28}$$

Evaluated at the origin, we find

$$A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} , \tag{29}$$

and so $\lambda_{1,2} = \pm i$. The origin is a $\underline{\underline{\text{center}}}$. The other fixed points are analyzed in the same manner:

$$A = \begin{pmatrix} 0 & 1 \\ 2 & 0 \end{pmatrix} , \tag{30}$$

and the eigenvalues are $\lambda_{1,2} = \pm\sqrt{2}$. $(\pm 1, 0)$ are $\underline{\underline{\text{saddle points}}}$.

b) The potential energy is given by

$$\frac{dV}{dx} = x - x^3 \, . \tag{31}$$

Integration yields

$$V(x) = \frac{1}{2}x^2 - \frac{1}{4}x^4 , \tag{32}$$

where we have set the integration constant to zero. The conserved quantity is then

$$\begin{aligned} E &= \frac{1}{2}\dot{x}^2 + V(x) \\ &= \underline{\underline{\frac{1}{2}\dot{x}^2 + \frac{1}{2}x^2 - \frac{1}{4}x^4}} . \end{aligned} \tag{33}$$

c) The phase portrait is shown in Fig. 8.

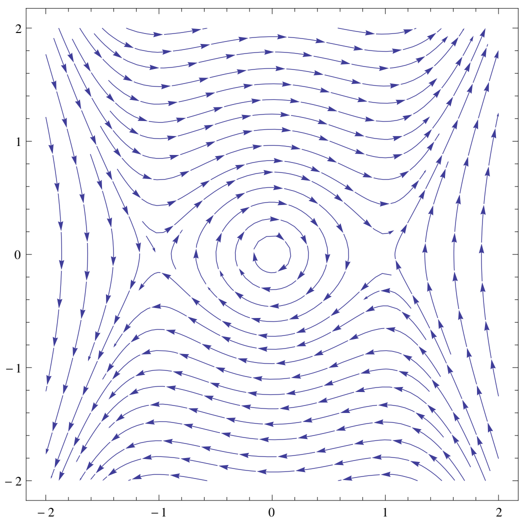

Figure 8: Phase portrait of problem 6.5.1.

Problem 6.6.3

a) The function $f(x,y) = \sin y$ is odd in y and the function $g(x,y) = \sin x$ is even in y. Hence the system is reversible.

b) The equations are

$$\dot{x} = \sin y , \tag{34}$$
$$\dot{y} = \sin x , \tag{35}$$
$$\tag{36}$$

The fixed points are $\underline{\underline{x^* = n\pi}}$ and $\underline{\underline{y^* = m\pi}}$, where n and m are integers. These points are the zeros of the sine function.

The Jacobian matrix is

$$A(x^*, y^*) \;=\; \begin{pmatrix} 0 & \cos y^* \\ \cos y^* & 0 \end{pmatrix} , \tag{37}$$

The eigenvalues are

$$\lambda_{1,2} \;=\; \pm\sqrt{\cos x^* \cos y^*} \; . \tag{38}$$

$\cos n\pi = 1$ if n even and $\cos n\pi = -1$ if n odd. This implies that the eigenvalues are $\lambda_{1,2} = \pm 1$ if n and m are even or if n and m are odd. The fix point is then a saddle. If n is even and m odd or vice versa, the eigenvalues are $\lambda = \pm i$. The fixed point is then a center.

c) We have

$$\begin{aligned} \frac{dy}{dx} &= \frac{\dot{y}}{\dot{x}} \\ &= \frac{\sin x}{\sin y} \; . \end{aligned} \tag{39}$$

At a point on the line $y = \pm x$, this ratio is ± 1 and so vector field points along the line itself. Hence, we stay on it.

d) The phase portrait and the type of fixed point can be seen in Fig. 9.

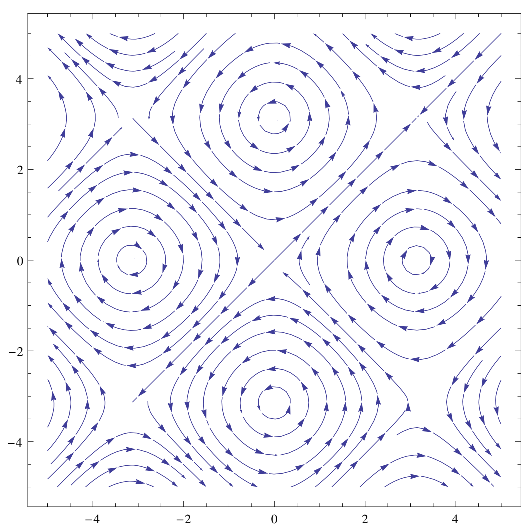

Figure 9: Phase portrait of problem 6.6.3.

Problem 6.6.10

The dynamics is given by the equations

$$\begin{aligned} \dot{x} &= -y - x^2 \, , && (40) \\ \dot{y} &= x \, . && (41) \end{aligned}$$

a) The only fixed point is the origin. The Jacobian matrix is given by

$$A(x,y) = \begin{pmatrix} 0-2x & -1 \\ 1 & 0 \end{pmatrix} . \qquad (42)$$

Evaluated at the origin, we find

$$A(0,0) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} , \qquad (43)$$

and so the eigenvalues are $\lambda = \pm i$. The origin is therefore a linear center. Note that the system in reversible since it is invariant under $t \to -t$ and $y \to -y$. Theorem 6.6.1 then applies and the origin is a nonlinear center. The phase portrait is shown in Fig. 10.

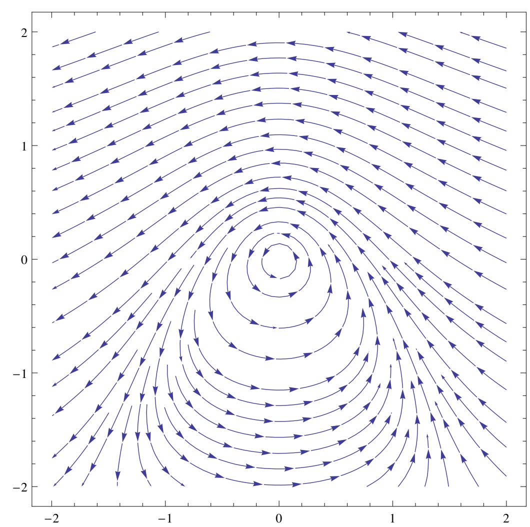

Figure 10: Phase portrait of problem 6.6.10

Problem 6.7.1

The equation for the damped pendulum reads

$$\ddot{\theta} + b\dot{\theta} + \sin\theta = 0 \, , \qquad (44)$$

where $b > 0$ is a parameter. Introducing $\nu = \dot{\theta}$, we can write the above equation as

$$\dot{\theta} = \nu , \tag{45}$$
$$\dot{\nu} = -b\nu - \sin\theta . \tag{46}$$

The fixed points are given by $\dot{\nu} = \dot{\theta} = 0$, i. e. $\nu = 0$ and therefore $\sin\theta = 0$. The latter yields $\theta = 0$ or $\theta = \pi$. The fixed points are therefore $\underline{\underline{(0,0) \text{ and } (\pi,0)}}$. The Jacobian matrix reads

$$A(\theta,\nu) = \begin{pmatrix} 0 & 1 \\ -\cos\theta & -b \end{pmatrix} \tag{47}$$

This gives the characteristic equation for the eigenvalues

$$\lambda(\lambda + b + \cos\theta) = 0 . \tag{48}$$

The solutions are

$$\lambda = \frac{-b \pm \sqrt{b^2 - 4\cos\theta}}{2} . \tag{49}$$

This yields $\tau = -b$ and $\Delta = \cos\theta$. For the fixed point $(0,0)$, this yields $\tau = -b$ and $\Delta = 1$. If $\tau^2 - 4\Delta = b^2 - 4 > 0$, i.e. if $b > 2$, the fixed point is a stable node, and if $b < 2$, the fixed point is a stable spiral. Moreover, For the fixed point $(0,\pi)$, this yields $\tau = -b$ and $\Delta = -1$. Thus the fixed point is a saddle point for all values of b.

Problem 7.1.4

The dynamics is governed by the equations

$$\dot{r} = r\sin r , \tag{50}$$
$$\dot{\theta} = 1 . \tag{51}$$

In Cartesian coordinates, this becomes

$$\dot{x} = x\sin\sqrt{x^2+y^2} - y , \tag{52}$$
$$\dot{y} = y\sin\sqrt{x^2+y^2} + x . \tag{53}$$

We note that $\dot{\theta} = 1$ everywhere and so the flow is always counterclockwise. The radial flow vanishes for $r = \pi, 2\pi$ etc and so we have circular orbits for these values of r. Between these values of r, the radial flow always has the same sign and it alternating. Between $r = 0$ and $r = \pi$, $\dot{r} > 0$ and so flow is away from the origin. Between $r = \pi$ and $r = 2\pi$, $\dot{r} < 0$ and so flow is towards the origin. The circular orbits are therefore limit cycles and their stability is alternating.

The phase portrait is shown in Fig. 11.

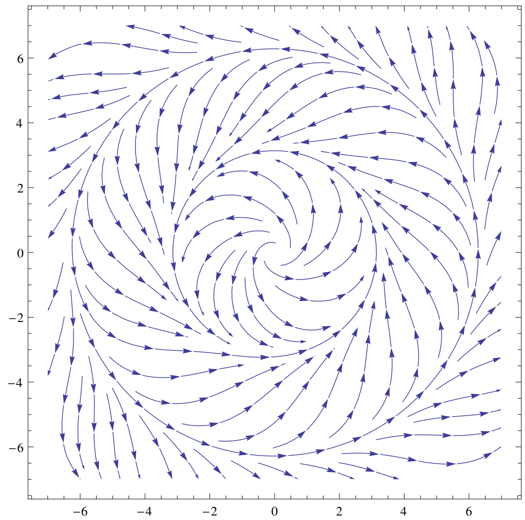

Figure 11: Phase portrait of problem 7.1.4.

Problem 8.1.9

The equation for the anharmonic oscillator with damping reads [1]

$$\ddot{x} + b\dot{x} - kx + x^3 \; = \; 0 \; . \tag{54}$$

This can be written as

$$\dot{x} \; = \; y \; , \tag{55}$$
$$\dot{y} \; = \; -by + kx - x^3 \; . \tag{56}$$

The fixed points are $\underline{\underline{(0,0)}}$ and $\underline{\underline{(\pm\sqrt{k},0)}}$. The Jacobian matrix reads

$$A(x,y) \; = \; \begin{pmatrix} 0 & 1 \\ k - 3x^2 & -b \end{pmatrix} \; . \tag{57}$$

This yields

$$A(0,0) \; = \; \begin{pmatrix} 0 & 1 \\ k & -b \end{pmatrix} \; , \tag{58}$$

and thus

$$\lambda \; = \; \frac{-b \pm \sqrt{b^2 + 4k}}{2} \; . \tag{59}$$

[1] Note that we have damping for $b > 0$. For $b < 0$ we are pumping energy into the system. Also note that the linear force term is the usual if $k < 0$. For $k > 0$ the force is no longer restoring but pushes the particle away from the origin. However, the cubic term stabilizes the system for all b and k.

Similarly

$$A(\pm\sqrt{k},0) \;=\; \begin{pmatrix} 0 & 1 \\ -2k & -b \end{pmatrix} , \tag{60}$$

and thus

$$\lambda \;=\; \frac{-b \pm \sqrt{b^2 - 8k}}{2} . \tag{61}$$

All fixed points are stable for $b > 0$ and unstable for $b < 0$. Whether it is a spiral or node depends on the sign of $b^2 + 4k$ and $b^2 - 8k$, respectively. For $b = 0$, there is no damping and the system is conservative. The potential is $V(x) = -\frac{1}{2}kx^2 + \frac{1}{4}x^4$. In this case, the origin is stable for $k < 0$ and unstable for $k > 0$. The other fixed points only exist for $k > 0$ and they are stable. This is the standard pitchfork bifurcation from chapter three. This is shown in Fig. 12

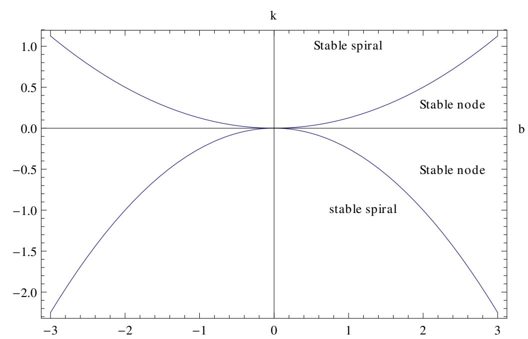

Figure 12: Stability diagram. Everything to the left is unstable.

Problem 8.2.8

The equations governing the dynamics are

$$\dot{x} \;=\; x[x(1-x) - y] , \tag{62}$$

$$\dot{y} \;=\; y(x-a) , \tag{63}$$

where $a \geq 0$ is a parameter.

a) The nullclines are given by $\dot{x} = 0$ and $\dot{y} = 0$ The former yields $\underline{\underline{x = 0}}$, and $\underline{\underline{y = x(1-x)}}$, while the latter gives $\underline{\underline{y = 0}}$ and $\underline{\underline{x = a}}$. This is shown Fig. 13 b) $\dot{y} = 0$ gives $y = 0$ or $x = a$. If $y = 0$, $\dot{x} = 0$ implies that $x = 0$ or $x = 1$. If $x = a$ $\dot{x} = 0$ implies $y = a(1-a)$. Thus the fixed points are $(x, y) = \underline{\underline{(0, 0)}}$, $(x, y) = \underline{\underline{(1, 0)}}$, and $(x, y) = \underline{\underline{(a, a(1-a))}}$. Note that last

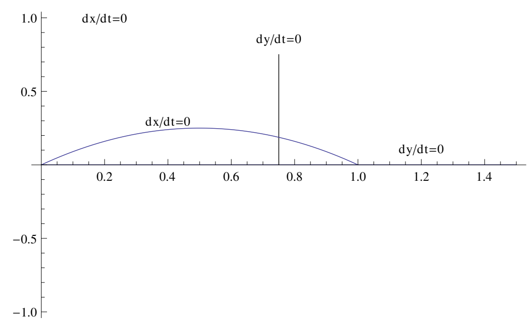

Figure 13: Nullclines for problem 8.2.8.

fixed point has $x < 0$ if $a < 0$ and $y < 0$ if $a > 1$ and so does not make sense biologically. Since $a \geq 0$ is specifies, we must restrict ourselves to $a \in [0, 1]$ for this fixed point.

The Jacobian matrix reads

$$A(x, y) \;=\; \begin{pmatrix} 2x - 3x^2 - y & -x \\ y & x - a \end{pmatrix} . \tag{64}$$

This yields

$$A(0, 0) \;=\; \begin{pmatrix} 0 & 0 \\ 0 & -a \end{pmatrix} , \tag{65}$$

and so the eigenvalues are $\lambda = 0$ and $\lambda = -a$. The origin is therefore marginally stable for all $a \geq 0$.

The phase portrait is shown Fig. 14 which shows that the origin is unstable. Moreover

$$A(1, 0) \;=\; \begin{pmatrix} -1 & -1 \\ 0 & 1 - a \end{pmatrix} , \tag{66}$$

and so the eigenvalues are $\lambda = -1$ and $\lambda = 1 - a$. The fixed point is therefore a saddle for $a < 1$ and a stable node for $a > 1$. For $a = 1$, the fixed point is marginally stable. The phase portrait is shown Fig. 14 indicating that the fixed point is actually stable.

Finally

$$A(a, a - a^2) \;=\; \begin{pmatrix} a - 2a^2 & -a \\ a - a^2 & 0 \end{pmatrix} , \tag{67}$$

with eigenvalues

$$\lambda \;=\; \frac{a - 2a^2 \pm a\sqrt{4a^2 - 3}}{2} . \tag{68}$$

For $a > 1/2$ it is a unstable and for $a < 1/2$ it is stable. For $4a^2 - 3 > 0$, i.e. for $a > \sqrt{3}/2$ it is a node and for $4a^2 - 3 < 0$, i.e. for $a < \sqrt{3}/2$ it is a spiral.

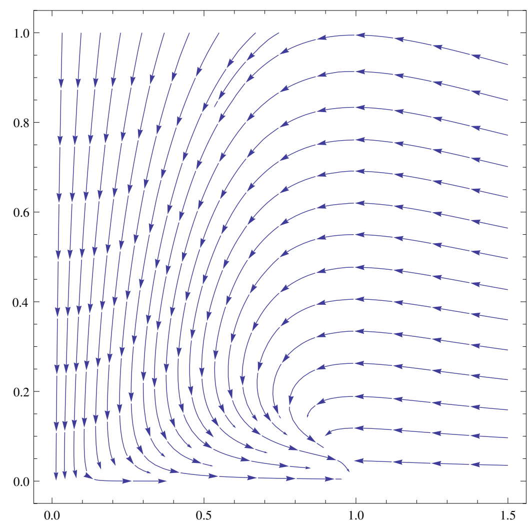

Figure 14: Phase portrait for $a = 1$ in problem 8.2.8.

c) The phase portrait is shown Fig. 15 which shows that you will flow to the line $y = 0$, i.e. the predators will go extinct. d) We have seen that the real part of the eigenvalues go through a zero at $a = a_c = 1/2$. Thus we have a Hopf bifurcation. The fixed point is losing stability and after that it is surrounded by a stable limit cycle (going towards lower values of a). It is therefore supercritical. The phase portrait for $a = 1/2$ is shown in Fig. 16. e)For $a = 1/2$, the imaginary part of the eigenvalue is $\omega = \sqrt{2}/4$.

f)

Problem 10.1.9

The map is

$$x_{n+1} = \frac{2x_n}{1 + x_n} . \tag{69}$$

The fixed points x are given by the equation

$$x = \frac{2x}{1 + x} . \tag{70}$$

This yields

$$x^2 + x = 2x , \tag{71}$$

or $\underline{\underline{x = 0}}$ and $\underline{\underline{x = 1}}$. The stability is given by the derivative of the function $f(x) = 2x/(1+x)$:

$$f'(x) = \frac{2}{(1 + x)^2} . \tag{72}$$

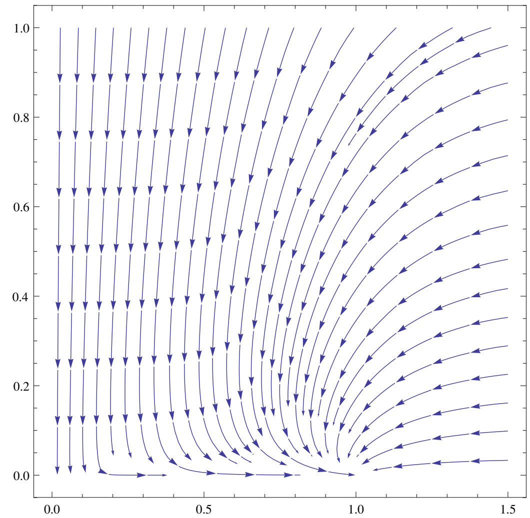

Figure 15: Phase portrait for $a = 2$ in problem 8.2.8.

This yields $f'(0) = 2$ and $f'(1) = \frac{1}{2}$. Hence the fixed point $x = 0$ is unstable and the fixed point $x = 1$ is stable.

Problem 10.1.10

The map is

$$x_{n+1} \quad = \quad 1 + \frac{1}{2}\sin(x_n)\,. \tag{73}$$

In Fig 17, we show the functions $g(x) = x$ and $h(x) = 1 + \frac{1}{2}\sin x$ from which it is evident that there is a unique fixed point that corresponds to the intersection of the curves.

The derivative of the function $f(x) = 1 + \frac{1}{2}\sin x$ is is

$$f'(x) = \frac{1}{2}\cos x\ . \tag{74}$$

Since $|f'(x)| \leq \frac{1}{2}$, the fixed point is stable.

Problem 10.1.12

Consider the map

$$f(x_n) \quad = \quad x_n - \frac{g(x_n)}{g'(x_n)}\ . \tag{75}$$

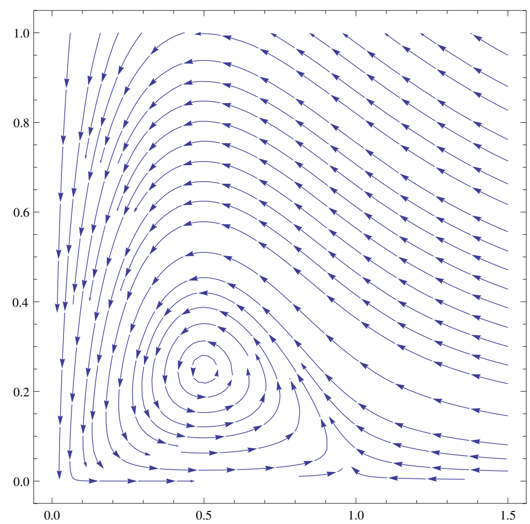

Figure 16: Phase portrait for $a = 1/2$ in problem 8.2.8.

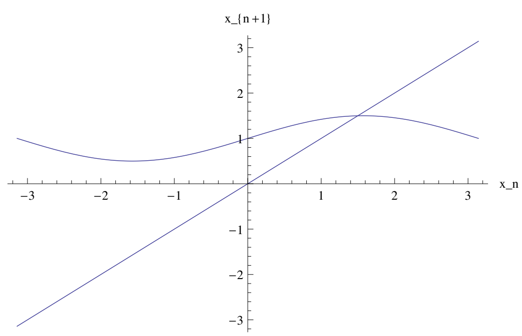

Figure 17: Fixed point of the map $x_{n+1} = 1 + \frac{1}{2}\sin(x_n)$.

a) For the function $g(x) = x^2 - 4$, this yields

$$f(x_n) \quad = \quad \underline{\underline{x_n - \frac{x_n^2 - 4}{2x_n}}} \; . \tag{76}$$

b) The fixed points are given by

$$x \quad = \quad x - \frac{x^2 - 4}{2x} \; . \tag{77}$$

This yields $x^2 - 4 = 0$ or $\underline{\underline{x = \pm 2}}$.

c) The derivative is

$$f'(x) \quad = \quad \frac{1}{2} - \frac{2}{x^2} \; , \tag{78}$$

and so

$$f'(\pm 2) = 0 . \tag{79}$$

This shows that the fixed points are supestable.

d) Try it out!

Problem 10.1.13

Differentiation of $f(x)$ gives

$$f'(x) = \frac{g(x)g''(x)}{[g'(x)]^2} . \tag{80}$$

Unless $g'(x) = 0$, $g(x) = 0$ corresponds to $f'(x) = 0$, i.e. a zero of $g(x)$ corresponds to a superstable fixed point for $f(x)$.

Problem 10.3.11

The map is given by

$$x_{n+1} = -(r+1)x - x^2 - 2x^3 . \tag{81}$$

a) Clearly $x = 0$ is fixed point. It stability is determined by the derivative of $f(x) = -(r+1)x - x^2 - 2x^3$:

$$f'(x) = -(r+1) - 2x - 6x^2 . \tag{82}$$

This yields $f'(0) = -1 - r$ and so the origin is stable for $\underline{\underline{-2 < r < 0}}$.

b) We notice that $f'(0) = -1$ for $r = 0$, which is the criterion for a flip bifurcation.

c) The period-2 cycles satisfy $f(f(x)) - x = 0$. Since $f(x) - x$ is a factor in this polynomial, we obtain by long division

$$[-(2+r)x - x^2 - 2x^3][-r + rx - (1 + 2r + 2r^2)x^2 - 4rx^3 - 2(3+4r)x^4 - 8x^5 - 8x^6] = 0 . \tag{83}$$

We are interested in a bifurcation for small negative values of r around $x = 0$. We can then approximate the second polynomial with

$$-r + rx - x^2 = 0 , \tag{84}$$

whose solutions are $x = (r \pm \sqrt{r^2 - 4r})/2 \approx= \pm\sqrt{-r}$. Thus the cycle exists for $r < 0$ and coalesces with $x = 0$ as $r \to 0^-$. The stability is determined by the derivative of $f(f(x))$ evaluated at the points of the cycle, i. e. by the chain rule $f'(\sqrt{-r})f'(-\sqrt{-r}) \approx (-1 - 2\sqrt{-r})(-1 + 2\sqrt{-r}) = 1 - 4r > 1$ and the 2-cycle is therefore unstable.

d) Use a cobweb to show that $x \to 0$ for $|x|$ small and $r < 0$ and $x \to \infty$ for $|x|$ small and $r > 0$.

Problem 10.4.2

The map is given by

$$x_{n+1} = \frac{rx_n^2}{1 + x_n^2} \ . \tag{85}$$

The origin is always a fixed point. The other fixed points are given by $x^2 - rx + 1$, or

$$x_\pm = \frac{r \pm \sqrt{r^2 - 4}}{2} \ . \tag{86}$$

These fixed points exist for $r \geq 2$. Moreover

$$f'(x) = \frac{2rx}{(1 + x^2)^2} \ , \tag{87}$$

which shows that the origin is always stable. At $r = 2$, the new fixed point is $x = 1$ and so $f'(1) = 1$. This shows that bifurcation is a tangent bifurcation. At the fixed point $x_\pm$, we have

$$f'(x_\pm) = \frac{2}{1 + x^2} \ , \tag{88}$$

which is always larger than zero. Thus there is no period-doubling. We can also prove this explicitly for a period-2 cycle. The existence of such a cycle implies $f[f(x)] = x$. Since the solutions to $f(x) = x$ are also solutions to $f[f(x)] = x$, we can use long division to write $f[f(x)] = x$ as

$$\frac{[(1 + x^2)(1 + x(r + x + r^2x))]}{[1 + 2x^2 + (1 + r^2)x^4)]} \ . = 0 \ . \tag{89}$$

The first term in the numerator has no solution. The second term yields $(r^2+1)x^2 - rx + 1 = 0$, which gives

$$x = -r \pm \sqrt{\frac{r^2 - 4(r^2 + 1)}{2(1 + r^2)}} \ . \tag{90}$$

which has no real solution, as anticipated. No chaos and no intermittency.

Problem 10.4.3

The map is given by

$$x_{n+1} \quad = \quad 1 - rx_n^2 \; . \tag{91}$$

A superstable n-cycle has by definition $[f^n(x)]' = 0$. Using the chain rule, this is equivalent to $f'(x) = 0$. In the present case, we have $f(x) = 1 - rx^2$ and so $f'(x) = 0$ yields $-2rx = 0$, i.e. $x = 0$. Moreover

$$f^3(x) \quad = \quad 1 - r[1 - r(1 - rx^2)^2]^2 \; . \tag{92}$$

The 3-cycle satisfies $f^3(x) = x$, where $x = 0$ is in the cycle. Inserting $x = 0$ into $f^3(x) = x$ gives

$$\underline{\underline{1 - r(1 - r)^2 = 0}} \; , \tag{93}$$

which is the sought polynomial.

Problem 10.5.4

The Lyapunov exponent is $\lambda = \log r$ and so chaos exists for $r > 1$. This implies there can be no periodic window $1 < r_1 < r < r_2$ after the onset of chaos.

Problem 10.7.6

Since R_2 is defined by superstability of the 4-cycle, we know that $x = 0$ is a point on that cycle. Thus $f^4(0, R_2) = 0$. This yields a polynomial for R_2:

$$R_2 - R_2^2 + 2R_2^3 - 5R_2^4 + 6R_2^5 - 6R_2^6 + 4R_2^7 - R_2^8 \quad = \quad 0 \; . \tag{94}$$

Since $R_2 - R_2^2$ is a factor of the above polynomial (since $R_2 - R_2^2 = 0$ defines the superstable 2-cycle), we find

$$1 + R_2^2(R_2 - 2)(R_2 - 1)(1 + R_2^2) \quad = \quad 0 \; . \tag{95}$$

There are roots $R_2 = 1.3107$ and $R_2 = 1.9408$. The function is plotted in Fig. 18.

Comparing $\alpha f^2(x/\alpha, R_1)$ and $\alpha f^4(x/\alpha^2, R_2)$ and matching coefficients through order x^2 with $R_1 = 1$ gives

$$R_2 - R_2^2 + 2R_2^3 - 5R_2^4 + 6R_2^5 - 6R_2^6 + 4R_2^7 - R_2^8 \quad = \quad 0 \; , \tag{96}$$

$$\alpha - 4R_2^3 + 8R_2^4 - 12R_2^5 + 12R_2^6 - 4R_2^7 \quad = \quad 0 \; . \tag{97}$$

The first equation is identical to Eq. (94). The second can be solved for α:

$$\alpha \quad = \quad 4R_2^3 - 8R_2^4 + 12R_2^5 - 12R_2^6 + 4R_2^7 \; . \tag{98}$$

Inserting $R_2 = 1.3107$ gives $\underline{\underline{\alpha = -2.44433}}$. The other root gives $\alpha = -20.3309$ and is not relevant. Note that if we had demanded the coefficient of x^4 to vanish, it would have given $\alpha = -2.334$.

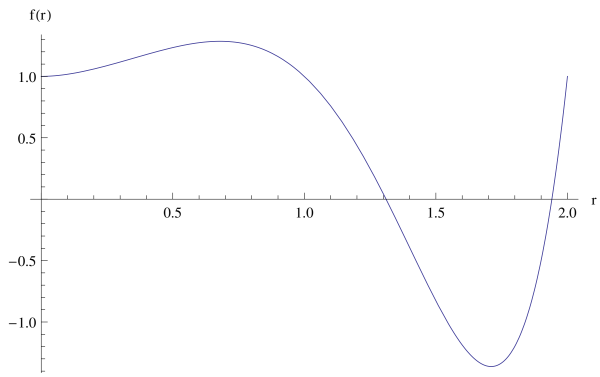

Figure 18: Polynomial for R_2.

Problem 11.1.6

The decimal shift map is given by

$$x_{n+1} \quad = \quad 10x_n \pmod 1 \; . \tag{99}$$

The map is shown is Fig. 19.

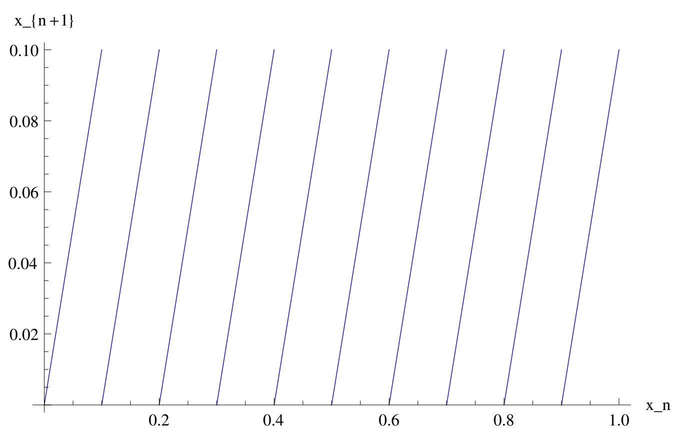

Figure 19: Decimal shift map.

a) A rational number y can be represented in decimal form

$$y \quad = \quad x_0.x_1x_2x_3.. \; , \tag{100}$$

where either the sequence of numbers is finite or it is periodic. If $y_o \in Q$, then we will eventually end up at $y = 0$ if the sequence is finite, or a periodic orbit given by the periodicity of the decimal representation. Thus a rational corresponds to a periodic orbit and Q is countable. Since the derivative of the decimal shift map is 10 (see Fig. 19), every orbit is unstable.

b) A irrational number has a aperiodic decimal representation and so if $y_0 \in R \backslash Q$ will give rise to an aperiodic orbit. The irrational numbers are uncountable.

c) A number on the form $x_0 = c_1 c_2 .... c_N c_N ...$, will after $n = N$ iterations be mapped to $x_n = c_N c_N .....$ After $n+1$ iterations we also have $x_{n+1} = c_N ...$ and so it is fixed point. Such a number will be periodic and therefore $x_0 \in Q$. The set is thus countable.

Problem 12.1.1

If either $|a| > 1$ or $|b| > 1$, the sequence $\{(x_n, y_n)\}$ will wander off to infinity. If $|a| = |b| = 1$, any (x_0, y_0) is a fixed point. If $|a| < 1$ and $|b| < 1$, the sequence $\{(x_n, y_n)\}$ converges to the origin. The way the sequence approaches the origin depends on the sign of a and b and in which quadrant you start i. e. of the point (x_0, y_0). For example if $0 < a1 <$ and $-1 < b < 0$ and (x_0, y_0) is in the first quadrant, the sequence will hop between the first and the third quadrant. Likewise if $0 < a < 1$ and $-1 < b < 0$ and (x_0, y_0) is in the second quadrant, the sequence hops between the second and third quadrant. The other cases, can be treated in the same way.

Problem 12.1.5

a) We can write (x_0, y_0) as $(.a_1 a_2 a_3 ..., .b_1 b_2 b_3 ...)$, where a_i and b_i are either 0 or 1. If $x_0 < \frac{1}{2}$, then $a_1 = 0$ and x_0 is multplied by two which corresponds to moving all digits one place to the left. Thus $x_1 = .a_2 a_3 ....$ Then y_0 is multiplied by a $\frac{1}{2}$ which corresponds to moving all digits one place to the right. This implies that $B(x, y) = (.a_2 a_3 ... 0 b_2 b_3 ...)$ If $x_0 \geq \frac{1}{2}$, then $a_1 = 1$. Multiplying by 2 *and* subtracting by one corresponds to moving all digits to the left and dropping a_1. Thus $x_1 = .a_2 a_3 ....$ Now y_0 is multiplied by a $\frac{1}{2}$ and we add $\frac{1}{2}$ The first operation is again moving all digits to the right while the second implies that the first digit will be one (since $\frac{1}{2=0.1}$ in the binary representation). Since $a_1 = 1$ we can write this as $y = .1 b_2 b_3 ... = .a_1 b_2 b_3 ...$ and so in both case we can write $\underline{\underline{B(x, y) = (.a_2 a_3 ... a_1 b_2 b_3 ...)}}$.

b) The only possibility for a period-2 orbit is that $x = 0.101010...$ or $x = 0.01010...$ since B moves the all digits to the left and $B^2(x, y) = (x, y)$. After two iteration of y_0, we know that $b_1 b_2$ has been replaced by $a_2 a_1$ and so $y = 0.010101$ is the only possibility for $x = 0.101010...$ and vice versa (use the same argument to determine $b_3 ...$). The number $x = 0.101010...$ equals

$$
\begin{aligned}
x &= \frac{1}{2} + \frac{1}{2^3} + ... \\
&= \frac{2}{3} \,. \qquad (101)
\end{aligned}
$$

Likewise, you can show that $x = 0.010101... = \frac{1}{3}$.

c) The only periodic orbits must be rational numbers because these numbers are either finite or periodic in the decimal representation and the action on x is to move all the digits to the left. y must also be rational, otherwise (x, y) cannot be periodic. In addition there are additional requirements analogous to those derived in b); $b_1 = a_n$, $b_2 = a_{n-1}$ etc for an n-cycle. Since the cartesian product $\mathbb{Q} \times \mathbb{Q}$ is countable, it follows.

d) Pick x irrational and y anything. x irrational implies that the orbit cannot be periodic since the action on x is to move all the digits to the left. Since the irrationals are uncoutable the results follows.