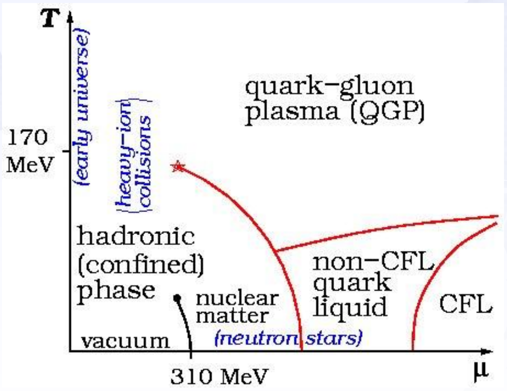
Spectral Analysis of dense 2-color QCD

Takuya Kanazawa

Norwegian Winter Workshop on QCD in Extreme Conditions

24 Feb 2010@Trondheim, Norway

- N. Yamamoto, TK, PRL**103**(2009) 032001 [arXiv:0902.4533]
- TK, T. Wettig, N. Yamamoto, JHEP**08**(2009)003 [arXiv:0906.3579]
- TK, T. Wettig, N. Yamamoto, [arXiv:0912.4999]



Aim of our study

To study the eigenvalues of **the Dirac operator**

in **2-color QCD** at high baryon density

using **Chiral random matrix theory** (ChRMT)

Aim of our study

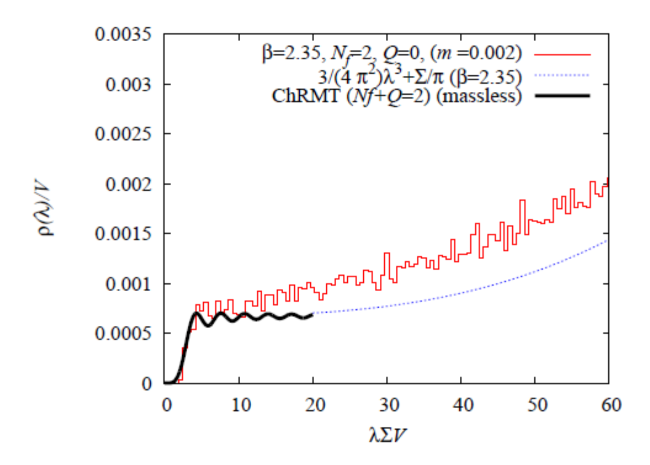
To study the eigenvalues of **the Dirac operator**

in **2-color QCD** at high baryon density

using **Chiral random matrix theory** (ChRMT)

- Phenomenological applications (Phase structure)
- Exact theory of Dirac eigenvalue distribution

From: JLQCD & TWQCD collaborations, [arXiv:0705.3322]



Motivation

Why Dirac eigenvalues?

--- Not directly observable

--- But related to important quantities such as the chiral condensate (by Banks-Casher)

$$|\langle \bar{q}q \rangle| = \frac{\pi \rho(0)}{V_4} \neq 0 , \qquad \rho(\lambda) \equiv \left\langle \sum_n \delta(\lambda - \lambda_n) \right\rangle$$

--- And to the Polyakov loop   Gattringer(06)

--- *Exact model-independent results* available (by equivalence of QCD and ChRMT in a specific limit)

QCD with Nc=2

- No sign problem at μ≠0 for even N_f and pairwise equal mass
  ➔ Monte Carlo Simulation feasible !

  Hands-Kogut-Lombardo-Morrison (99)
- Color-singlet diquarks (baryonic pion)
  ∵Pauli-Gursey Symmetry
- Chiral Symmetry Breaking at μ~0: $SU(2N_f) \to Sp(2N_f)$
- Arena of diverse theoretical approaches:

  ChPT, NJL, PNJL, RMM, SCE on the lattice, …
- Rich physics on the phase diagram:

  Superfluid phase transition, BEC-BCS crossover, FFLO phase, vector meson condensation, …
- Close resemblance to *Nc=3 QCD at finite isospin density*

High-density limit $(T \simeq 0)$

- Asymptotic free Fermi surface
- Attractive channel: $[2]_C \times [2]_C = [3]_C + [1]_C$

  → Cooper instability → BCS gap

Barrois(77)
Bailin-Love(84)

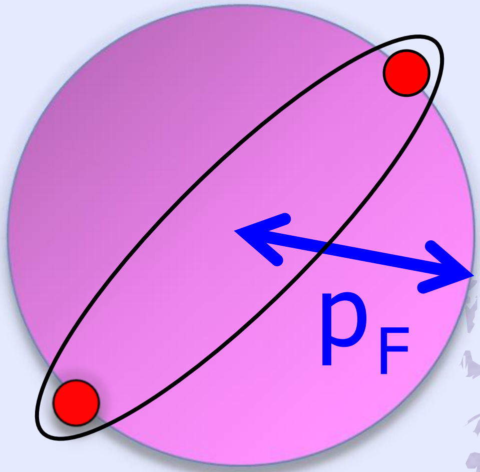
$$\Lambda_{SU(2)} \ll \Delta \sim \mu g^{-5} e^{-\frac{2\pi^2}{g}} \ll \mu$$

Son(98)

- χ SB by $\langle qq \rangle$ ← BCS superfluid

$$\mathrm{SU}(N_f)_R \times \mathrm{SU}(N_f)_L \times \mathrm{U}(1)_B \times \mathrm{U}(1)_A \rightarrow \mathrm{Sp}(N_f)_R \times \mathrm{Sp}(N_f)_L$$

- Chiral Lagrangian valid at energy scales below Δ?



↔ Kogut-Stephanov-Toublan-Verbaarschot-Zhitnitzky(00)

High-density chiral Lagrangian

TK-Wettig-Yamamoto(09)

$(N_f \geq 4$, even)

$$\Sigma_{R,L} \in \mathrm{SU}(N_f)/\mathrm{Sp}(N_f), \qquad V, A \in \mathrm{U}(1)$$

$$\begin{aligned}\mathcal{L}_{\mathrm{NG}} = {} & \frac{f_H^2}{2}\Big\{|\partial_0 V|^2 - v_H^2|\partial_i V|^2\Big\} + \frac{f_{\eta'}^2}{2}\Big\{|\partial_0 A|^2 - v_{\eta'}^2|\partial_i A|^2\Big\} \\ & + \frac{f_\pi^2}{2}\,\mathrm{tr}\Big\{|\partial_0 \Sigma_L|^2 - v_\pi^2|\partial_i \Sigma_L|^2 + (L \leftrightarrow R)\Big\} \\ & - \frac{3}{4\pi^2}\Delta^2\Big\{A^2\,\mathrm{tr}\,(M\Sigma_R M^T \Sigma_L^\dagger) + \mathrm{c.c.}\Big\} + (\text{sub-leading terms})\,.\end{aligned}$$

- M^2 acts as a source for Δ^2.
- Modified GOR relation: $m_\pi \propto \dfrac{\Delta}{\mu} M$.

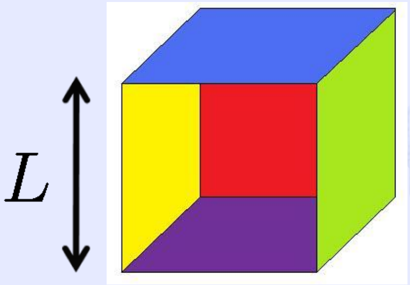
(Likewise for CFL phase: Casalbuoni-Gatto(99) Son-Stephanov(00))

The ε-regime at high density

Gasser-Leutwyler (87)
Leutwyler-Smilga(92)
TK-Yamamoto(09)

2-color QCD in a finite volume with

$$\frac{1}{\Delta} \ll L \ll \frac{1}{m_\pi}$$



$V_4 \equiv L^4$

'Dimensional reduction'

↓

$$Z(M) = \int_{U(1)_A} dA \int_{SU(N_f)_L/Sp(N_f)_L} d\Sigma_L \int_{SU(N_f)_R/Sp(N_f)_R} d\Sigma_R \exp\left[V_4 \frac{3\Delta^2}{4\pi^2}\left\{A^2 \mathrm{Tr}\,(M\Sigma_R M^T \Sigma_L^\dagger) + \mathrm{c.c.}\right\}\right]$$

$$\propto \frac{1}{\prod_{i>j}(m_i^2 - m_j^2)} \mathrm{Pf}\left[(m_j^2 - m_i^2) I_0\left(\frac{3}{\pi^2} V_4 \Delta^2 m_i m_j\right)\right]_{1\le i,j\le N_f}.$$

<u>Partition function does **not** depend on μ explicitly.</u>

Taylor expansion in M:

$$Z(M) = 1 + \frac{2}{(N_f-1)^2}\left(V_4 \frac{3\Delta^2}{4\pi^2}\right)^2 \left\{(\mathrm{Tr}\, M^\dagger M)^2 - \mathrm{Tr}\,(M^\dagger M)^2\right\} + O(M^8).$$

Comparison with at each order $Z_{\rm QCD}(M) = \left\langle \prod_n{}' \det\left(1 + \frac{M^\dagger M}{\lambda_n^2}\right)\right\rangle$

→ Spectral sum rules

$$\left\langle \sum_n \frac{1}{\lambda_n^2} \right\rangle = \left\langle \sum_{m\neq n} \frac{1}{\lambda_m^2 \lambda_n^2} \right\rangle = \left\langle \sum_n \frac{1}{\lambda_n^6} \right\rangle = 0, \qquad \left\langle \sum_n \frac{1}{\lambda_n^4} \right\rangle = \frac{9}{2(N_f-1)^2\pi^4}(V_4\Delta^2)^2, \quad \ldots$$

Microscopic spectral function

$$\int_{\mathbb{C}} d^2z \frac{\rho_s(z)}{z^4} = \frac{9}{2(N_f-1)^2\pi^4} \quad \text{where} \quad \begin{cases} \rho_s(z) \equiv \lim_{V_4\to\infty} \frac{1}{V_4\Delta^2}\rho\left(\frac{z}{\sqrt{V_4\Delta^2}}\right) \\ \rho(\lambda) \equiv \left\langle \sum_n \delta^2(\lambda-\lambda_n)\right\rangle \end{cases}$$

- Dirac eigenvalues of order $O\left(\frac{1}{\sqrt{V_4\Delta^2}}\right)$ are govened by Δ.
- Can we obtain $\rho_s(z)$ explicitly?

What is the ChRMT for this ε-regime?

$\left(\text{cf. } O\left(\frac{1}{V_4\Sigma}\right) \text{ at } \mu = 0\right)$

Known ChRMTs at μ≠0

Osborn(04)
Akemann et al.(04)

- Two-matrix formulation

$$Z = \int dA\, dB \prod_{f=1}^{N_f} \det \begin{pmatrix} \hat{m}_f & iA + \mu B \\ iA^\dagger + \mu B^\dagger & \hat{m}_f \end{pmatrix} e^{-N\alpha^2(\mathrm{tr}\, A^\dagger A + \mathrm{tr}\, B^\dagger B)}$$

- Vanderheyden-Jackson model

Vanderheyden-Jackson(00,01)

$$Z(\mu,T) = \int \mathcal{D}H\, \mathcal{D}\psi_1^\dagger\, \mathcal{D}\psi_1\, \mathcal{D}\psi_2^T\, \mathcal{D}\psi_2^* \times \exp\left[i \begin{pmatrix} \psi_1^\dagger \\ \psi_2^T \end{pmatrix}^T \begin{pmatrix} \mathcal{H} + (\pi T + i\mu)\gamma_0 + im & \eta P_\Delta \\ -\eta^* P_\Delta & -\mathcal{H}^T + (\pi T - i\mu)\gamma_0^T - im \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2^* \end{pmatrix} \right].$$

The μ-term dominates the matrix at $\mu \to \infty$;

→ Neither of them support <qq> at arbitrarily large density

("No Fermi sea")

A candidate chRMT for dense 2-color QCD

TK-Wettig-Yamamoto(arXiv:0912.4999)

$$Z(\{m_f\}) = \int dA dB \prod_{f=1}^{N_f} \det \begin{pmatrix} \hat{m}_f \mathbf{1} & A \\ B & \hat{m}_f \mathbf{1} \end{pmatrix} \mathrm{e}^{-N\alpha^2 \mathrm{tr}\,(AA^T + BB^T)}$$

[A, B: real-valued $N \times N$ matrices]

= Check list =

- ✓ Dirac operator should be chiral, real, and non-Hermitian
- ✓ The correct SSB pattern
- ✓ μ does not appear in the Dirac operator at all
- ✓ It should generate the spectral sum rules resulting from the chiral Lagrangian

A candidate chRMT for dense 2-color QCD

TK-Wettig-Yamamoto(arXiv:0912.4999)

$$Z(\{m_f\}) = \int dAdB \prod_{f=1}^{N_f} \det \begin{pmatrix} \hat{m}_f \mathbf{1} & A \\ B & \hat{m}_f \mathbf{1} \end{pmatrix} \mathrm{e}^{-N\alpha^2 \mathrm{tr}\,(AA^T + BB^T)}$$

[A, B: real-valued $N \times N$ matrices]

Correct SSB pattern? ---Yes.

$$\det{}^{N_f} \begin{pmatrix} 0 & A \\ B & 0 \end{pmatrix} = \det{}^{N_f/2} \begin{pmatrix} 0 & A \\ -A^T & 0 \end{pmatrix} \det{}^{N_f/2} \begin{pmatrix} 0 & B \\ -B^T & 0 \end{pmatrix}$$

$\mathrm{SU}(N_f) \to \mathrm{Sp}(N_f)$ $\mathrm{SU}(N_f) \to \mathrm{Sp}(N_f)$

$$N \gg 1$$

Hubbard-Stratonovich + Saddle point method

$$Z(\hat{M}) \sim \int_{\mathrm{U}(N_f)} dU\, dV \;\mathrm{Pf}^{N}\begin{pmatrix} \frac{1}{\sqrt{2}\alpha}(VIV^T)^\dagger & -\hat{M}^* \\ \hat{M}^\dagger & \frac{1}{\sqrt{2}\alpha}UIU^T \end{pmatrix} \mathrm{Pf}^{N}\begin{pmatrix} \frac{1}{\sqrt{2}\alpha}(UIU^T)^\dagger & -\hat{M}^T \\ \hat{M} & \frac{1}{\sqrt{2}\alpha}VIV^T \end{pmatrix}$$

$$\sim \int_{\mathrm{U}(N_f)} dU\, dV \;\exp\left[-2N\alpha^2 \mathrm{Re\,tr}\,(\hat{M}UIU^T\hat{M}^T V^* I V^\dagger)\right]$$

$$= \int_{\mathrm{U}(N_f)/Sp(N_f)} d\Sigma_R \int_{\mathrm{U}(N_f)/Sp(N_f)} d\Sigma_L \;\exp\left[-2N\alpha^2 \mathrm{Re\,tr}\,(\hat{M}\Sigma_R\hat{M}^T\Sigma_L^\dagger)\right]$$

Identical to the ε-regime chiral Lagrangian, with $N\alpha^2\hat{M}^2 \leftrightarrow \frac{3}{4\pi^2}V_4\Delta^2 M^2$

➔ All sum rules are recovered

What if a diquark source is added?

$(N_f = 2)$

$$Z = \int d[A,B,u,d,\bar{u},\bar{d}]\ \exp\left[\begin{pmatrix}\bar{u}_R\\ \bar{u}_L\\ d_R\\ d_L\end{pmatrix}\left(\begin{array}{cc|cc} m_u & A & & \boldsymbol{j}^* \\ B & m_u & \boldsymbol{j}^* & \\ \hline & \boldsymbol{j} & m_d & -A^T \\ \boldsymbol{j} & & -B^T & m_d \end{array}\right)\begin{pmatrix}u_L\\ u_R\\ \bar{d}_L\\ \bar{d}_R\end{pmatrix}\right]$$

$$\times \exp\left[-N\alpha^2 \,\mathrm{tr}\,(AA^T + BB^T)\right]$$

$$\sim \oint d\phi_{L,R}\ \exp\left[2\sqrt{2}N\alpha|\boldsymbol{j}|(\cos\phi_L - \cos\phi_R) - 4N\alpha^2 m_u m_d \cos(\phi_L - \phi_R)\right]$$

$$\sim \exp\left[4\sqrt{2}N\alpha|\boldsymbol{j}| + 4N\alpha^2 m_u m_d\right].$$

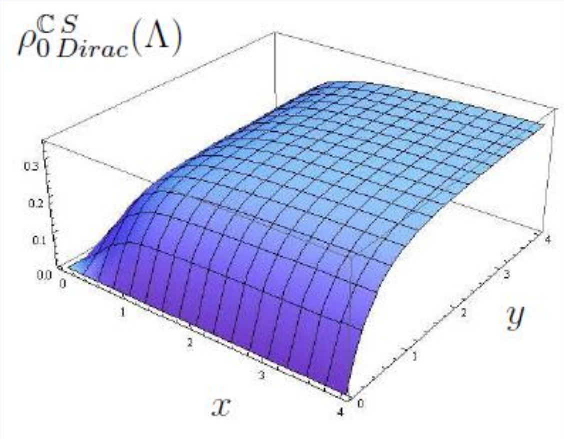
✓Non-zero diquark condensate
✓Zero chiral condensate

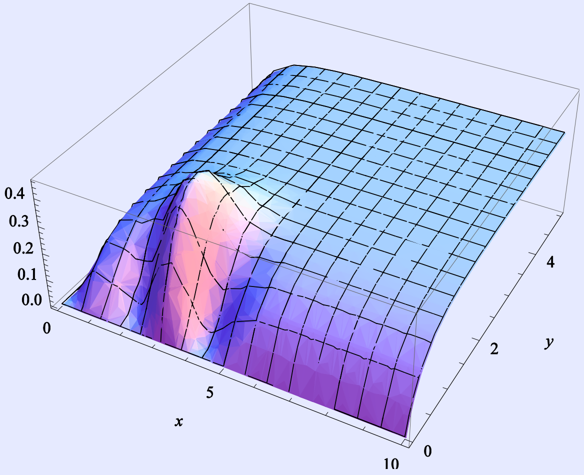
Likewise for $N_f \geq 4$

Microscopic Spectral Density of complex eigenvalues

$$\rho_s(z) := \lim_{N\to\infty} \frac{1}{N\alpha^2}\rho\left(\frac{z}{\sqrt{N\alpha^2}}\right) \quad \text{with } \rho(\lambda) := \left\langle \sum_{n=1}^{2N} \delta^2(\lambda - \lambda_n) \right\rangle$$

$N_f = 0$





$N_f = 2$

TK-Wettig-Yamamoto, in preparation

Left figure: from Akemann, Phillips, Sommers, [arXiv:0911.1276]

Relation to the model in the literature

Akemann-Phillips-Sommers(07)

$$Z_\nu(\hat{\mu}, \{\hat{m}_f\}) = \int dC\, dD\ \ \mathrm{e}^{-2N\alpha^2 \,\mathrm{tr}\,(CC^T + DD^T)} \times \prod_{f=1}^{N_f} \det \begin{pmatrix} \hat{m}_f \mathbf{1} & C + \hat{\mu} D \\ -C^T + \hat{\mu} D^T & \hat{m}_f \mathbf{1} \end{pmatrix}.$$

・・・ Introduced as the chRMT for 2-color QCD at **low** density

$$N \to \infty \quad \text{with} \quad \boxed{N \hat{m}_f} \ \text{and} \ \boxed{N \hat{\mu}^2} \ \text{fixed}$$

Our model is obtained from this model as a special case: $\boxed{\widehat{\mu} = 1, \nu = 0}$

A single model describes two extremes!

Low density limit $\widehat{\mu} = O(1/\sqrt{N})$	High density limit $\widehat{\mu} = 1$
$N\alpha\widehat{M} \leftrightarrow V_4\Sigma M$ $N\widehat{\mu}^2 \leftrightarrow V_4 F_\pi^2 \mu^2$	$N\alpha^2\widehat{M}^2 \leftrightarrow V_4\Delta^2 M^2$

Summary&Outlook

1. We proposed a new chRMT for the high-density limit of 2-color QCD
2. The first application of strongly non-Hermitian RMT to QCD $(\hat{\mu} \not\to 0 \text{ as } N \to \infty)$
3. Future directions:

Extension to intermediate μ ?

Adjoint QCD? Odd *Nf* ?

Check by Monte Carlo? Sign problem?

Spectral density with diquark source?

=END=