NJL-model studies of the QCD phase structure

TECHNISCHE UNIVERSITÄT DARMSTADT

Michael Buballa

Norwegian Winter Workshop on QCD in Extreme Conditions

Trondheim, Norway, February 24 - 26, 2010

The Nambu–Jona-Lasinio model

TECHNISCHE UNIVERSITÄT DARMSTADT

PHYSICAL REVIEW VOLUME 122, NUMBER 1 APRIL 1, 1961

Dynamical Model of Elementary Particles Based on an Analogy with Superconductivity. I*

Y. NAMBU AND G. JONA-LASINIO†
The Enrico Fermi Institute for Nuclear Studies and the Department of Physics, The University of Chicago, Chicago, Illinois

(Received October 27, 1960)

It is suggested that the nucleon mass arises largely as a self-energy of some primary fermion field through the same mechanism as the appearance of energy gap in the theory of superconductivity. The idea can be put into a mathematical formulation utilizing a generalized Hartree-Fock approximation which regards real nucleons as quasi-particle excitations. We consider a simplified model of nonlinear four-fermion interaction which allows a γ_5-gauge group. An interesting consequence of the symmetry is that there arise automatically pseudoscalar zero-mass bound states of nucleon-antinucleon pair which may be regarded as an idealized pion. In addition, massive bound states of nucleon number zero and two are predicted in a simple approximation.
The theory contains two parameters which can be explicitly related to observed nucleon mass and the pion-nucleon coupling constant. Some paradoxical aspects of the theory in connection with the γ_5 transformation are discussed in detail.

- two papers almost 50 years ago: Phys. Rev. **122**, 345-358; ibid. **124**, 246-254 (1961).
- cited more than 3200 (1450) times
- Nambu: Nobel prize in physics 2008

  "for the discovery of the mechanism of spontaneous broken symmetry in subatomic physics"
- Nobel lecture presented by Jona-Lasinio

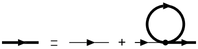
  http://nobelprize.org/nobel_prizes/physics/laureates/2008/nambu-lecture.html

NJL model: main ideas and results of the original papers

- Lagrangian: $\mathcal{L} = \bar{\psi}(i\partial\!\!\!/ - m)\psi + G\left[(\bar{\psi}\psi)^2 + (\bar{\psi}i\gamma_5\vec{\tau}\psi)^2\right]$
  - 4-point interaction, invariant under chiral transformations $\psi \to \exp(i\vec{\theta}\cdot\vec{\tau}\gamma_5)\psi$
  - chiral symmetry explicitly broken by (small) fermion mass m

NJL model: main ideas and results of the original papers

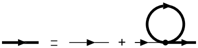
- **Lagrangian:** $\mathcal{L} = \bar{\psi}(i\partial\!\!\!/ - m)\psi + G\left[(\bar{\psi}\psi)^2 + (\bar{\psi}i\gamma_5\vec{\tau}\psi)^2\right]$
  - 4-point interaction, invariant under chiral transformations $\psi \to \exp(i\vec{\theta}\cdot\vec{\tau}\gamma_5)\psi$
  - chiral symmetry explicitly broken by (small) fermion mass m
- **spontaneous symmetry breaking:** $\langle\bar{\psi}\psi\rangle \neq 0$



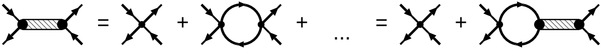
  - dynamical generation of a “constituent mass” $M \gg m$
  - lowers the ground-state energy of the system

NJL model: main ideas and results of the original papers

- Lagrangian: $\mathcal{L} = \bar\psi(i\partial\!\!\!/ - m)\psi + G\left[(\bar\psi\psi)^2 + (\bar\psi i\gamma_5\vec\tau\psi)^2\right]$
  - 4-point interaction, invariant under chiral transformations $\psi \to \exp(i\vec\theta\cdot\vec\tau\gamma_5)\psi$
  - chiral symmetry explicitly broken by (small) fermion mass m
- spontaneous symmetry breaking: $\langle\bar\psi\psi\rangle \neq 0$



  - dynamical generation of a "constituent mass" $M \gg m$
  - lowers the ground-state energy of the system
- mesonic excitations:



  - massless pions in the chiral limit (➜ Goldstone theorem, 1961)
  - $m_\pi^2 \propto m$ (➜ Gell-Mann–Oakes–Renner relation, 1968)

Later developments: brief history of the NJL model

- reinterpretation in the QCD era: schematic model for quarks

  [H. Kleinert, Erice lectures (1976); M.K. Volkov, Annals Phys. (1984); T. Hatsuda, T. Kunihiro, PLB (1984); ...]

  - problem: no confinement (➜ e.g., $q\bar{q}$-decays of mesons!)

Later developments: brief history of the NJL model

- reinterpretation in the QCD era: schematic model for quarks

  [H. Kleinert, Erice lectures (1976); M.K. Volkov, Annals Phys. (1984); T. Hatsuda, T. Kunihiro, PLB (1984); ...]

  - problem: no confinement (➜ e.g., $q\bar{q}$-decays of mesons!)

- nonzero temperature and density

  [V. Bernard, U.-G. Meißner, I. Zahed, PRD (1987); T. Hatsuda, T. Kunihiro, PLB (1987); ...]

Later developments: brief history of the NJL model

- reinterpretation in the QCD era: schematic model for quarks

  [H. Kleinert, Erice lectures (1976); M.K. Volkov, Annals Phys. (1984); T. Hatsuda, T. Kunihiro, PLB (1984); ...]

  - problem: no confinement (➜ e.g., $q\bar{q}$-decays of mesons!)
- nonzero temperature and density

  [V. Bernard, U.-G. Meißner, I. Zahed, PRD (1987); T. Hatsuda, T. Kunihiro, PLB (1987); ...]
- three quark flavors

  [D. Ebert, H. Reinhardt, NPB (1986) V. Bernard, R.L. Jaffe, U.-G. Meißner, PLB (1987); T. Hatsuda, T. Kunihiro, PLB (1987); ...]

  - nondegenerate strange quark mass
  - 6-point interaction to model the $U_A(1)$ anomaly (➜ $\eta - \eta'$ mass splitting)

Later developments: brief history of the NJL model

TECHNISCHE UNIVERSITÄT DARMSTADT

- reinterpretation in the QCD era: schematic model for quarks

  [H. Kleinert, Erice lectures (1976); M.K. Volkov, Annals Phys. (1984); T. Hatsuda, T. Kunihiro, PLB (1984); ...]

  - problem: no confinement (➜ e.g., $q\bar{q}$-decays of mesons!)
- nonzero temperature and density

  [V. Bernard, U.-G. Meißner, I. Zahed, PRD (1987); T. Hatsuda, T. Kunihiro, PLB (1987); ...]
- three quark flavors

  [D. Ebert, H. Reinhardt, NPB (1986) V. Bernard, R.L. Jaffe, U.-G. Meißner, PLB (1987); T. Hatsuda, T. Kunihiro, PLB (1987); ...]

  - nondegenerate strange quark mass
  - 6-point interaction to model the $U_A(1)$ anomaly (➜ $\eta - \eta'$ mass splitting)
- color superconductivity

  [M. Alford, K. Rajagopal, F. Wilczek, PLB (1998); R. Rapp, T. Schäfer, E.V. Shuryak, M. Velkovsky, PRL (1998); ...]

Later developments: brief history of the NJL model

TECHNISCHE UNIVERSITÄT DARMSTADT

- reinterpretation in the QCD era: schematic model for quarks

  [H. Kleinert, Erice lectures (1976); M.K. Volkov, Annals Phys. (1984); T. Hatsuda, T. Kunihiro, PLB (1984); ...]
  - problem: no confinement (➜ e.g., $q\bar{q}$-decays of mesons!)
- nonzero temperature and density

  [V. Bernard, U.-G. Meißner, I. Zahed, PRD (1987); T. Hatsuda, T. Kunihiro, PLB (1987); ...]
- three quark flavors

  [D. Ebert, H. Reinhardt, NPB (1986) V. Bernard, R.L. Jaffe, U.-G. Meißner, PLB (1987); T. Hatsuda, T. Kunihiro, PLB (1987); ...]
  - nondegenerate strange quark mass
  - 6-point interaction to model the $U_A(1)$ anomaly (➜ $\eta - \eta'$ mass splitting)
- color superconductivity

  [M. Alford, K. Rajagopal, F. Wilczek, PLB (1998); R. Rapp, T. Schäfer, E.V. Shuryak, M. Velkovsky, PRL (1998); ...]
- Polyakov-loop enhanced NJL model

  [K. Fukushima, PLB (2004); E. Megías, E. Ruiz Arriola, L. L. Salcedo,PRD (2006), C. Ratti, M.A. Thaler, W. Weise, PRD (2006); ...]
  - “statistical realization” of confinement

Critical questions and remarks

- Since we have QCD, why should we care about a model?

  (It seems, Nambu and Jona-Lasinio didn't find it worthwhile to write more than two papers on it.)

Critical questions and remarks

TECHNISCHE UNIVERSITÄT DARMSTADT

- Since we have QCD, why should we care about a model?

  (It seems, Nambu and Jona-Lasinio didn't find it worthwhile to write more than two papers on it.)
- Often NJL-model calculations are much simpler than QCD calculations.
  But can we trust the results?
  - In the best case, the results agree with model-independent theorems, but then we know them anyway.
  - Model-dependent results could be different from QCD.

Critical questions and remarks

- Since we have QCD, why should we care about a model?

  (It seems, Nambu and Jona-Lasinio didn't find it worthwhile to write more than two papers on it.)
- Often NJL-model calculations are much simpler than QCD calculations. But can we trust the results?
  - In the best case, the results agree with model-independent theorems, but then we know them anyway.
  - Model-dependent results could be different from QCD.
- Drawbacks of the model:
  - non-renormalizable

    ➜ results depend on cutoff parameters and the employed regularization scheme, and there are usually cutoff artifacts
  - no confinement
  - chirally symmetric, but symmetries do not uniquely fix the interaction

    ➜ many possible interaction terms, many parameters
  - temperature and density dependence of the effectice couplings unknown and usually neglected

...and some positive answers

- Sometimes models can help to identify yet unknown model-independent theorems (e.g., Goldstone's theorem and GOR relation after the original NJL paper) .

. . . and some positive answers

- Sometimes models can help to identify yet unknown model-independent theorems (e.g., Goldstone's theorem and GOR relation after the original NJL paper) .
- "Model-independent" predictions are often based on rather unphysical assumptions

  (e.g., Taylor expansions in "small parameters" which are not really small or not constant) .

  ➜ Models could help to identify situations where these predictions may fail.

...and some positive answers

- Sometimes models can help to identify yet unknown model-independent theorems (e.g., Goldstone's theorem and GOR relation after the original NJL paper) .
- "Model-independent" predictions are often based on rather unphysical assumptions

  (e.g., Taylor expansions in "small parameters" which are not really small or not constant) .

  ➜ Models could help to identify situations where these predictions may fail.
- Models can be employed for simplified explorative studies
  - to identify interesting problems, which should then be studied more seriously (e.g., the existence of a critical endpoint in the QCD phase diagram)
  - to test ideas and techniques used in other frameworks (e.g., methods to find the critical endpoint in lattice QCD) .

. . . and some positive answers

- Sometimes models can help to identify yet unknown model-independent theorems (e.g., Goldstone's theorem and GOR relation after the original NJL paper) .
- "Model-independent" predictions are often based on rather unphysical assumptions
  (e.g., Taylor expansions in "small parameters" which are not really small or not constant) .
  ➜ Models could help to identify situations where these predictions may fail.
- Models can be employed for simplified explorative studies
  - to identify interesting problems, which should then be studied more seriously
    (e.g., the existence of a critical endpoint in the QCD phase diagram)
  - to test ideas and techniques used in other frameworks
    (e.g., methods to find the critical endpoint in lattice QCD) .
- But we should always keep the limitations in mind and know when to stop ...

Outline

TECHNISCHE UNIVERSITÄT DARMSTADT

1. Introduction ✔
2. The NJL phase diagram at nonzero temperature and density
3. Color superconductivity
4. Inhomogeneous phases
5. The PNJL model

TECHNISCHE UNIVERSITÄT DARMSTADT

NJL PHASE DIAGRAM

Thermodynamics of the NJL model: mean-field approximation

TECHNISCHE UNIVERSITÄT DARMSTADT

- Lagrangian:

$$\mathcal{L} = \bar{q}(i\partial\!\!\!/ - m)q + G\left[(\bar{q}q)^2 + (\bar{q}i\gamma_5\vec{\tau}q)^2\right]$$

Thermodynamics of the NJL model: mean-field approximation

- Lagrangian:

$$\mathcal{L} = \bar{q}(i\partial\!\!\!/ - m)q + G\left[(\bar{q}q)^2 + (\bar{q}i\gamma_5\vec{\tau}q)^2\right]$$

- bosonize:

$$\mathcal{L} = \bar{q}\left(i\partial\!\!\!/ - m + 2G(\sigma + i\gamma_5\vec{\tau}\cdot\vec{\pi})\right)q - G\left(\sigma^2 + \vec{\pi}^{\,2}\right)$$

where, by the equations of motion, $\sigma = \bar{q}q, \quad \vec{\pi} = \bar{q}i\gamma_5\vec{\tau}q$

Thermodynamics of the NJL model: mean-field approximation

TECHNISCHE UNIVERSITÄT DARMSTADT

- Lagrangian:

$$\mathcal{L} = \bar{q}(i\partial\!\!\!/ - m)q + G\left[(\bar{q}q)^2 + (\bar{q}i\gamma_5\vec{\tau}q)^2\right]$$

- bosonize:

$$\mathcal{L} = \bar{q}\left(i\partial\!\!\!/ - m + 2G(\sigma + i\gamma_5\vec{\tau}\cdot\vec{\pi})\right)q - G\left(\sigma^2 + \vec{\pi}^{\,2}\right)$$

where, by the equations of motion, $\sigma = \bar{q}q, \quad \vec{\pi} = \bar{q}i\gamma_5\vec{\tau}q$

- constant mean fields: $\sigma(x) = \phi = const., \quad \pi_a(x) = 0$

➜ mean-field Lagrangian:

$$\mathcal{L}_{MF} \;=\; \bar{q}(i\partial\!\!\!/ - m + 2G\phi)q - G\phi^2 \;\equiv\; \mathcal{L}_M \;-\; \mathcal{V}_M$$

with

$\mathcal{L}_M = \bar{q}(i\partial\!\!\!/ - M)q$ free fermion with mass $M = m - 2G\phi$

$\mathcal{V}_M = G\phi^2 = \frac{(M-m)^2}{4G}$ field independent "potential"

Thermodynamics of the NJL model: thermodynamic potential

- thermodynamic potential per volume: $\Omega(T,\mu) = -\frac{T}{V} \ln \mathcal{Z}$

Thermodynamics of the NJL model: thermodynamic potential

- thermodynamic potential per volume: $\Omega(T,\mu) = -\frac{T}{V}\ln \mathcal{Z}$
- mean-field: $\mathcal{L}_{MF} = \mathcal{L}_M - \mathcal{V}_M$

$$\Rightarrow \quad \Omega_{MF}(T,\mu;M) = \Omega_M(T,\mu) + \mathcal{V}_M$$

$$= -12\int \frac{d^3p}{(2\pi)^3}\left\{E_p + T\ln\left(1+\exp\left(-\frac{E_p-\mu}{T}\right)\right) + T\ln\left(1+\exp\left(-\frac{E_p+\mu}{T}\right)\right)\right\} + \frac{(M-m)^2}{4G}$$

Thermodynamics of the NJL model: thermodynamic potential

TECHNISCHE UNIVERSITÄT DARMSTADT

- thermodynamic potential per volume: $\Omega(T,\mu) = -\frac{T}{V} \ln \mathcal{Z}$
- mean-field: $\mathcal{L}_{MF} = \mathcal{L}_M - \mathcal{V}_M$

$$\Rightarrow \quad \Omega_{MF}(T,\mu;M) = \Omega_M(T,\mu) + \mathcal{V}_M$$

$$= -12 \int \frac{d^3p}{(2\pi)^3} \Big\{ E_p + T \ln\Big(1 + \exp(-\frac{E_p - \mu}{T})\Big) + T \ln\Big(1 + \exp(-\frac{E_p + \mu}{T})\Big) \Big\} + \frac{(M-m)^2}{4G}$$

  - general bilinear Lagrangian:

$$\mathcal{L}_{bil} = \bar{q}\, S^{-1}\, q \quad \Rightarrow \quad \Omega_{bil} = -\frac{T}{V}\, \mathbf{Tr} \ln \frac{S^{-1}}{T} = -T \sum_n \int \frac{d^3p}{(2\pi)^3}\, \mathrm{Tr} \ln \Big(\frac{1}{T} S^{-1}(i\omega_n, \vec{p}) \Big)$$

Thermodynamics of the NJL model: thermodynamic potential

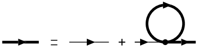
- thermodynamic potential per volume: $\Omega(T,\mu) = -\frac{T}{V} \ln \mathcal{Z}$
- mean-field: $\mathcal{L}_{MF} = \mathcal{L}_M - \mathcal{V}_M$

$$\Rightarrow \quad \Omega_{MF}(T,\mu;M) = \Omega_M(T,\mu) + \mathcal{V}_M$$

$$= -12 \int \frac{d^3p}{(2\pi)^3} \Big\{ E_p + T \ln\Big(1 + \exp(-\frac{E_p - \mu}{T})\Big) + T \ln\Big(1 + \exp(-\frac{E_p + \mu}{T})\Big)\Big\} + \frac{(M-m)^2}{4G}$$

  - general bilinear Lagrangian:

    $\mathcal{L}_{bil} = \bar{q}\, S^{-1}\, q \quad \Rightarrow \quad \Omega_{bil} = -\frac{T}{V}\, \mathbf{Tr} \ln \frac{S^{-1}}{T} = -T \sum_n \int \frac{d^3p}{(2\pi)^3} \,\mathrm{Tr}\, \ln\Big(\frac{1}{T} S^{-1}(i\omega_n, \vec{p})\Big)$

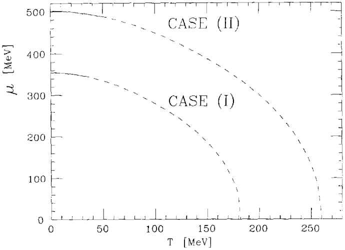
- stable solution: minimize Ω_{MF} w.r.t. M ➜ $M = M(T,\mu)$
  - $\frac{\partial \Omega_{MF}}{\partial M} = 0$ ➜ Hartree gap equation: 

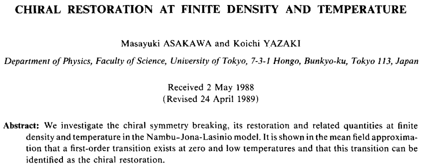
Phase diagram

TECHNISCHE UNIVERSITÄT DARMSTADT

- first NJL phase diagram:

  [M. Asakawa, K. Yazaki, NPA (1989)]





CHIRAL RESTORATION AT FINITE DENSITY AND TEMPERATURE

Masayuki ASAKAWA and Koichi YAZAKI

Department of Physics, Faculty of Science, University of Tokyo, 7-3-1 Hongo, Bunkyo-ku, Tokyo 113, Japan

Received 2 May 1988
(Revised 24 April 1989)

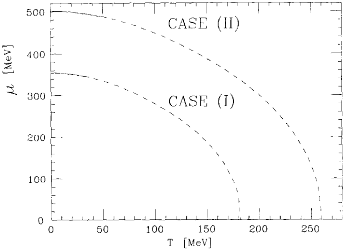
Abstract: We investigate the chiral symmetry breaking, its restoration and related quantities at finite density and temperature in the Nambu–Jona-Lasinio model. It is shown in the mean field approximation that a first-order transition exists at zero and low temperatures and that this transition can be identified as the chiral restoration.

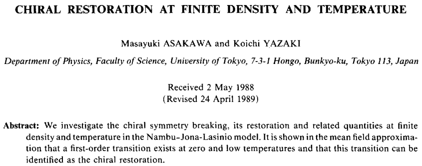
Phase diagram

TECHNISCHE UNIVERSITÄT DARMSTADT

- first NJL phase diagram:

  [M. Asakawa, K. Yazaki, NPA (1989)]





CHIRAL RESTORATION AT FINITE DENSITY AND TEMPERATURE

Masayuki ASAKAWA and Koichi YAZAKI

Department of Physics, Faculty of Science, University of Tokyo, 7-3-1 Hongo, Bunkyo-ku, Tokyo 113, Japan

Received 2 May 1988
(Revised 24 April 1989)

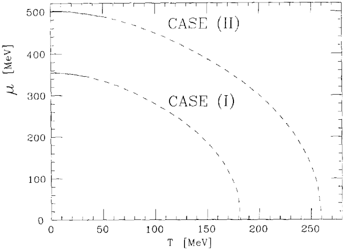
Abstract: We investigate the chiral symmetry breaking, its restoration and related quantities at finite density and temperature in the Nambu–Jona-Lasinio model. It is shown in the mean field approximation that a first-order transition exists at zero and low temperatures and that this transition can be identified as the chiral restoration.

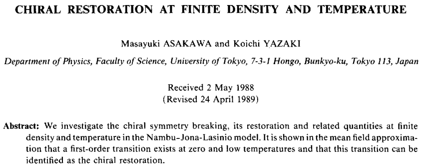
- first-order phase transition at low T and large μ, cross-over at high T and low μ ➜ critical endpoint !

Phase diagram

- first NJL phase diagram:

[M. Asakawa, K. Yazaki, NPA (1989)]





CHIRAL RESTORATION AT FINITE DENSITY AND TEMPERATURE

Masayuki ASAKAWA and Koichi YAZAKI

Department of Physics, Faculty of Science, University of Tokyo, 7-3-1 Hongo, Bunkyo-ku, Tokyo 113, Japan

Received 2 May 1988
(Revised 24 April 1989)

Abstract: We investigate the chiral symmetry breaking, its restoration and related quantities at finite density and temperature in the Nambu–Jona-Lasinio model. It is shown in the mean field approximation that a first-order transition exists at zero and low temperatures and that this transition can be identified as the chiral restoration.

- first-order phase transition at low T and large μ, cross-over at high T and low μ ➜ critical endpoint !
- location depends on parameter choice

Influence of vector interactions

TECHNISCHE UNIVERSITÄT DARMSTADT

- include vector interaction: $\mathcal{L}_V = -G_V(\bar{q}\gamma^\mu q)^2$

Influence of vector interactions

TECHNISCHE
UNIVERSITÄT
DARMSTADT

- include vector interaction: $\mathcal{L}_V = -G_V(\bar{q}\gamma^\mu q)^2$
- mean field: $\langle \bar{q}\gamma^\mu q \rangle = n\, g^{\mu 0}$ (quark number density)

➜ $\Omega_{MF}(T,\mu;M,\tilde{\mu}) = \Omega_M(T,\tilde{\mu}) + \frac{(M-m)^2}{4G} - \frac{(\mu-\tilde{\mu})^2}{4G_V}, \qquad \tilde{\mu} = \mu - 2G_V\, n$

Influence of vector interactions

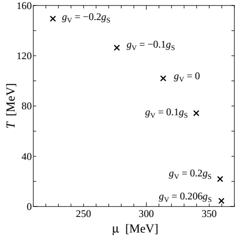
TECHNISCHE UNIVERSITÄT DARMSTADT

- include vector interaction: $\mathcal{L}_V = -G_V(\bar{q}\gamma^\mu q)^2$
- mean field: $\langle\bar{q}\gamma^\mu q\rangle = n\, g^{\mu 0}$ (quark number density)

➜ $\Omega_{MF}(T,\mu;M,\tilde{\mu}) = \Omega_M(T,\tilde{\mu}) + \frac{(M-m)^2}{4G} - \frac{(\mu-\tilde{\mu})^2}{4G_V}, \qquad \tilde{\mu} = \mu - 2G_V\, n$

- location of the CEP (PNJL):

[K. Fukushima, PRD (2008)]



- Positive (negative) G_V weaken (strengthen) the first-order phase transition.
- The CEP can be shifted around.
- Fit to ω mass in vacuum: $G_V \sim G$

  ➜ no first order!
- Can we trust the fit at high density?

Isospin chemical potential

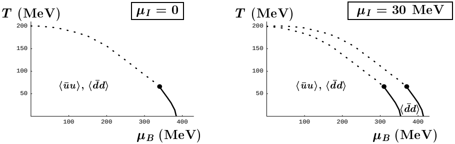
TECHNISCHE
UNIVERSITÄT
DARMSTADT

- unequal chemical potentials: $\mu_u = \mu + \delta\mu, \quad \mu_u = \mu - \delta\mu$

Isospin chemical potential

- unequal chemical potentials: $\mu_u = \mu + \delta\mu, \quad \mu_u = \mu - \delta\mu$
- phase diagram:

[D. Toublan, J.B. Kogut, PLB (2003)]

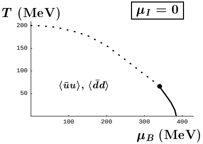


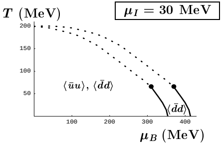
Isospin chemical potential

TECHNISCHE UNIVERSITÄT DARMSTADT

- unequal chemical potentials: $\mu_u = \mu + \delta\mu, \quad \mu_u = \mu - \delta\mu$
- phase diagram:

[D. Toublan, J.B. Kogut, PLB (2003)]





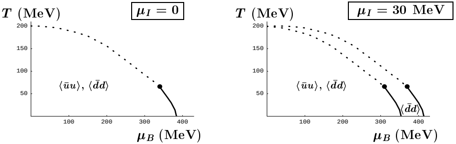
- generalized interaction: $\mathcal{L}_{int} = \mathcal{L}_1 + \mathcal{L}_2$
  - $\mathcal{L}_1 = (1-\alpha)\, G\left[(\bar{q}q)^2 + (\bar{q}\,\vec{\tau}q)^2 + (\bar{q}\,i\gamma_5 q)^2 + (\bar{q}\,i\gamma_5\vec{\tau}q)^2\right]$ ($U(2)_L \times U(2)_R$ symm.)
  - $\mathcal{L}_2 = \alpha\, G\left[(\bar{q}q)^2 - (\bar{q}\,\vec{\tau}q)^2 - (\bar{q}\,i\gamma_5 q)^2 + (\bar{q}\,i\gamma_5\vec{\tau}q)^2\right]$ ($U_A(1)$ breaking)

Isospin chemical potential

TECHNISCHE UNIVERSITÄT DARMSTADT

- unequal chemical potentials: $\mu_u = \mu + \delta\mu, \quad \mu_u = \mu - \delta\mu$
- phase diagram:

[D. Toublan, J.B. Kogut, PLB (2003)]

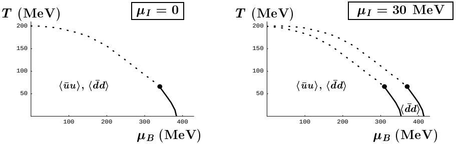


- generalized interaction: $\mathcal{L}_{int} = \mathcal{L}_1 + \mathcal{L}_2$
  - $\mathcal{L}_1 = (1-\alpha)\, G \left[(\bar{q}q)^2 + (\bar{q}\,\vec{\tau}q)^2 + (\bar{q}\,i\gamma_5 q)^2 + (\bar{q}\,i\gamma_5\vec{\tau}q)^2 \right]$ $\quad (U(2)_L \times U(2)_R$ symm.)
  - $\mathcal{L}_2 = \alpha\, G \left[(\bar{q}q)^2 - (\bar{q}\,\vec{\tau}q)^2 - (\bar{q}\,i\gamma_5 q)^2 + (\bar{q}\,i\gamma_5\vec{\tau}q)^2 \right]$ $\quad (U_A(1)$ breaking)
  - $\Omega = \Omega_{M_u}(T,\mu_u) + 2\,G\,\phi_u^2 + \Omega_{M_d}(T,\mu_d) + 2\,G\,\phi_d^2 - 2\,G\,\alpha(\phi_u - \phi_d)^2$

Isospin chemical potential

- unequal chemical potentials: $\mu_u = \mu + \delta\mu, \quad \mu_u = \mu - \delta\mu$
- phase diagram:

[D. Toublan, J.B. Kogut, PLB (2003)]



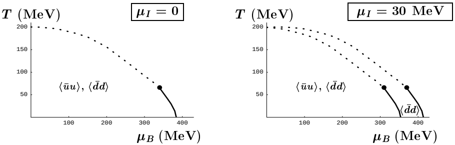
- generalized interaction: $\mathcal{L}_{int} = \mathcal{L}_1 + \mathcal{L}_2$
  - $\mathcal{L}_1 = (1-\alpha)\, G \left[(\bar{q}q)^2 + (\bar{q}\,\vec{\tau}q)^2 + (\bar{q}\,i\gamma_5 q)^2 + (\bar{q}\,i\gamma_5\vec{\tau}q)^2\right]$ ($U(2)_L \times U(2)_R$ symm.)
  - $\mathcal{L}_2 = \alpha\, G \left[(\bar{q}q)^2 - (\bar{q}\,\vec{\tau}q)^2 - (\bar{q}\,i\gamma_5 q)^2 + (\bar{q}\,i\gamma_5\vec{\tau}q)^2\right]$ ($U_A(1)$ breaking)
  - $\Omega = \Omega_{M_u}(T,\mu_u) + 2\,G\,\phi_u^2 + \Omega_{M_d}(T,\mu_d) + 2\,G\,\phi_d^2 - 2\,G\,\alpha(\phi_u - \phi_d)^2$
  - standard NJL: $\alpha = 0.5$, Toublan & Kogut: $\alpha = 0$ ➜ flavors decouple

Isospin chemical potential

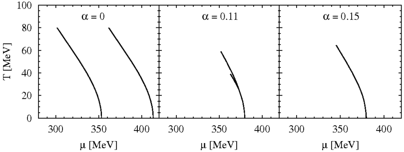
TECHNISCHE UNIVERSITÄT DARMSTADT

- unequal chemical potentials: $\mu_u = \mu + \delta\mu, \quad \mu_u = \mu - \delta\mu$
- phase diagram:

[D. Toublan, J.B. Kogut, PLB (2003)]



[M. Frank, M.B., M. Oertel, PLB (2003)]



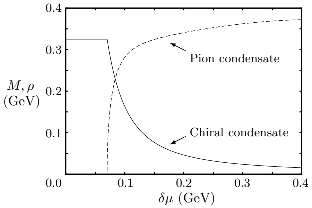
- generalized interaction: $\mathcal{L}_{int} = \mathcal{L}_1 + \mathcal{L}_2$
  - $\mathcal{L}_1 = (1-\alpha)\, G \left[(\bar q q)^2 + (\bar q\, \vec\tau q)^2 + (\bar q\, i\gamma_5 q)^2 + (\bar q\, i\gamma_5 \vec\tau q)^2\right]$ $\quad (U(2)_L \times U(2)_R$ symm.)
  - $\mathcal{L}_2 = \alpha\, G \left[(\bar q q)^2 - (\bar q\, \vec\tau q)^2 - (\bar q\, i\gamma_5 q)^2 + (\bar q\, i\gamma_5 \vec\tau q)^2\right]$ $\quad (U_A(1)$ breaking)
  - $\Omega = \Omega_{M_u}(T, \mu_u) + 2\,G\,\phi_u^2 + \Omega_{M_d}(T, \mu_d) + 2\,G\,\phi_d^2 - 2\,G\,\alpha(\phi_u - \phi_d)^2$
  - standard NJL: $\alpha = 0.5$, Toublan & Kogut: $\alpha = 0$ ➜ flavors decouple

Pion condensation

TECHNISCHE UNIVERSITÄT DARMSTADT

- charged pion condensation: $\langle \bar{q}\, i\gamma_5 \tau_1 q \rangle \neq 0$
  - favored for $|\mu_u - \mu_d| > m_\pi$ at $T = 0$
  - melts at higher T

[J.O. Andersen, L. Kyllingstad, JPG (2010)]

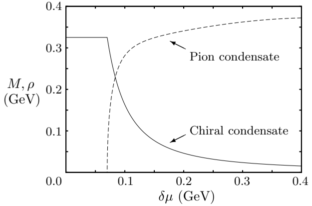


Pion condensation

TECHNISCHE UNIVERSITÄT DARMSTADT

- charged pion condensation: $\langle \bar{q}\, i\gamma_5 \tau_1 q \rangle \neq 0$
  - favored for $|\mu_u - \mu_d| > m_\pi$ at $T = 0$
  - melts at higher T

[J.O. Andersen, L. Kyllingstad, JPG (2010)]



- phase diagram in the $\mu_u - \mu_d$ plane:

[H.J. Warringa,D. Boer, J.O. Andersen, PRD (2005)]

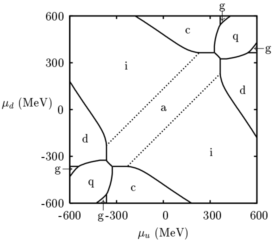
a: $\langle \bar{u}u \rangle, \langle \bar{d}d \rangle \neq 0$

c: $\langle \bar{u}u \rangle \neq 0$

d: $\langle \bar{d}d \rangle \neq 0$

i: $\pi^{\pm}$ condensation

q: color superconducting (2SC)



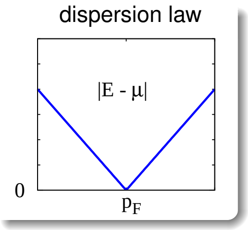
COLOR SUPERCONDUCTIVITY

BCS pairing

TECHNISCHE UNIVERSITÄT DARMSTADT

- Cooper instability:

  Fermi gas



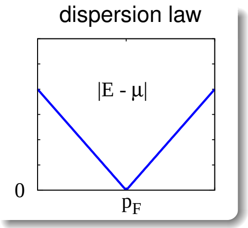
BCS pairing

TECHNISCHE UNIVERSITÄT DARMSTADT

- Cooper instability:

  Fermi gas + attraction

  ➜ condensation of Cooper pairs



BCS pairing

TECHNISCHE UNIVERSITÄT DARMSTADT

- Cooper instability:

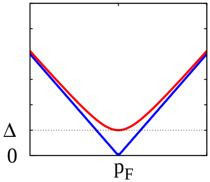
  Fermi gas + attraction

  ➜ condensation of Cooper pairs

  ➜ reorganisation of the Fermi surface

  ➜ gaps

dispersion law



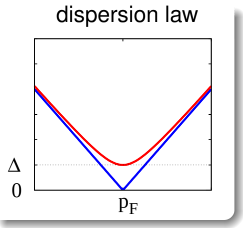
BCS pairing

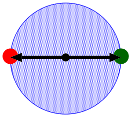
TECHNISCHE UNIVERSITÄT DARMSTADT

- Cooper instability:

  Fermi gas + attraction
  ➜ condensation of Cooper pairs
  ➜ reorganisation of the Fermi surface
  ➜ gaps

- Cooper pairs in BCS theory:
  - close to the Fermi surface
  - opposite momenta





BCS pairing

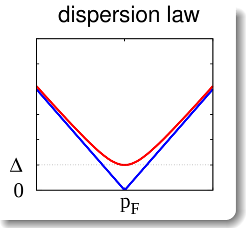
- Cooper instability:

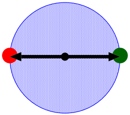
  Fermi gas + attraction

  ➜ condensation of Cooper pairs

  ➜ reorganisation of the Fermi surface

  ➜ gaps





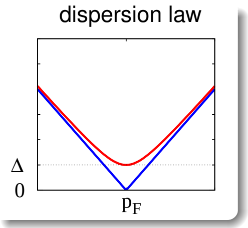
- Cooper pairs in BCS theory:
  - close to the Fermi surface
  - opposite momenta

  ➜ works only if $p_F^a = p_F^b$

BCS pairing

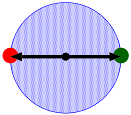
- Cooper instability:

  Fermi gas + attraction
  ➜ condensation of Cooper pairs
  ➜ reorganisation of the Fermi surface
  ➜ gaps

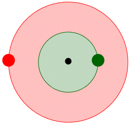


- Cooper pairs in BCS theory:
  - close to the Fermi surface
  - opposite momenta

  ➜ works only if $p_F^a = p_F^b$



- unequal Fermi momenta: $p_F^{a,b} = \bar{p}_F \pm \delta p_F$

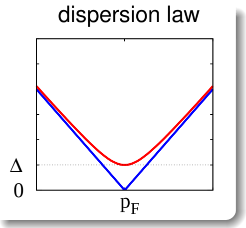


BCS pairing

TECHNISCHE UNIVERSITÄT DARMSTADT

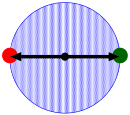
- Cooper instability:

  Fermi gas + attraction
  ➜ condensation of Cooper pairs
  ➜ reorganisation of the Fermi surface
  ➜ gaps

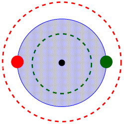


- Cooper pairs in BCS theory:
  - close to the Fermi surface
  - opposite momenta

  ➜ works only if $p_F^a = p_F^b$



- unequal Fermi momenta: $p_F^{a,b} = \bar{p}_F \pm \delta p_F$
  - first equalize Fermi momenta

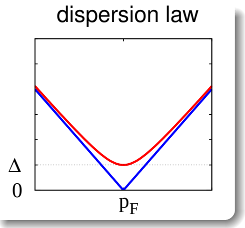


BCS pairing

TECHNISCHE UNIVERSITÄT DARMSTADT

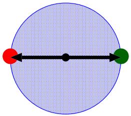
- Cooper instability:

  Fermi gas + attraction
  ➜ condensation of Cooper pairs
  ➜ reorganisation of the Fermi surface
  ➜ gaps

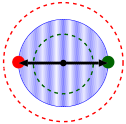


- Cooper pairs in BCS theory:
  - close to the Fermi surface
  - opposite momenta

  ➜ works only if $p_F^a = p_F^b$



- unequal Fermi momenta: $p_F^{a,b} = \bar{p}_F \pm \delta p_F$
  - first equalize Fermi momenta
  - then pair



BCS pairing

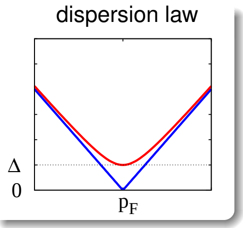
- Cooper instability:

  Fermi gas + attraction

  ➜ condensation of Cooper pairs

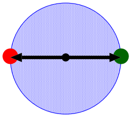
  ➜ reorganisation of the Fermi surface

  ➜ gaps

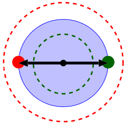


- Cooper pairs in BCS theory:
  - close to the Fermi surface
  - opposite momenta

  ➜ works only if $p_F^a = p_F^b$



- unequal Fermi momenta: $p_F^{a,b} = \bar{p}_F \pm \delta p_F$
  - first equalize Fermi momenta
  - then pair
  - only favored if $\delta p_F \lesssim \frac{\Delta}{\sqrt{2}}$ [Chandrasekhar, Clogston (1962)]



Diquark condensates

TECHNISCHE UNIVERSITÄT DARMSTADT

- QCD: attractive interaction among quarks
  - ➜ diquark condensates: $\langle q^T \mathcal{O} q \rangle$

    $\mathcal{O}$ = operator (totally antisymmetric)

Diquark condensates

- QCD: attractive interaction among quarks

  → diquark condensates: $\langle q^T \mathcal{O}\, q \rangle$

  $\mathcal{O}$ = operator (totally antisymmetric)

- most important example:

$$\langle q^T C\gamma_5\, \tau_2\, \lambda_2\, q \rangle \;\sim\; \underbrace{(\uparrow\downarrow - \downarrow\uparrow)}_{\text{spin}} \otimes \underbrace{(ud - du)}_{\text{flavor}} \otimes \underbrace{(r\,g - g\,r)}_{\text{color}}$$

  - spin 0, antisymmetric in color and flavor
  - dominant for one-gluon exchange, instantons ...

NJL model for color superconductivity

TECHNISCHE UNIVERSITÄT DARMSTADT

- include quark-quark interaction terms, e.g.,

$$\mathcal{L}_{qq} = H\,(\bar{q}\,i\gamma_5\tau_A\lambda_{A'}\,C\bar{q}^T)(q^T C\,i\gamma_5\tau_A\lambda_{A'}\,q)$$

  - natural extension of the NJL model
    (which was “based on an analogy with superconductivity”)
  - can in principle be obtained by Fierz transformation from a $\bar{q}q$-interaction

NJL model for color superconductivity

TECHNISCHE UNIVERSITÄT DARMSTADT

- include quark-quark interaction terms, e.g.,

$$\mathcal{L}_{qq} = H\,(\bar{q}\,i\gamma_5\tau_A\lambda_{A'}\,C\bar{q}^T)(q^T C\,i\gamma_5\tau_A\lambda_{A'}\,q)$$

  - natural extension of the NJL model
    (which was "based on an analogy with superconductivity")
  - can in principle be obtained by Fierz transformation from a $\bar{q}q$-interaction

- mean-field approximation: $\langle q^T\,C\gamma_5\,\tau_2\,\lambda_2\,q\rangle = -2H\Delta$
  - Nambu-Gor'kov bispinors: $\Psi = \frac{1}{\sqrt{2}}\begin{pmatrix} q \\ C\bar{q}^T \end{pmatrix}$
  - mean-field Lagrangian:

$$\mathcal{L}_{MF} + \mu\,q^\dagger q = \bar{\Psi}\begin{pmatrix} i\partial\!\!\!/ + \mu\gamma^0 & \Delta\,\gamma_5\tau_2\lambda_2 \\ -\Delta^*\,\gamma_5\tau_2\lambda_2 & -i\overleftarrow{\partial\!\!\!/} - \mu\gamma^0 \end{pmatrix}\Psi - \frac{|\Delta|^2}{4H}$$

  - still bilinear, but with a more complicated inverse propagator!

Competing condensates

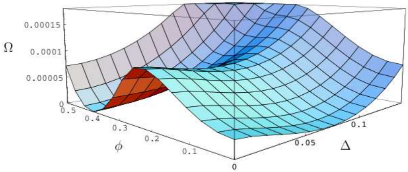
TECHNISCHE UNIVERSITÄT DARMSTADT

- include both: $\bar{q}q$ and qq interactions ➜ competition $\langle \bar{q}q \rangle$ vs. Δ

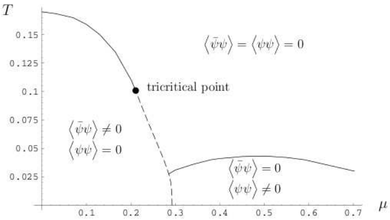
  [J. Berges, K. Rajagopal, NPB (1999)]

thermodynamic potential:

($T = 0$, $\mu = \mu_c$)



phase diagram (chiral limit):



three-flavor systems

TECHNISCHE UNIVERSITÄT DARMSTADT

- antisymmetric flavor combinations:

  $\Delta_{ud} \sim (ud - du), \qquad \Delta_{us} \sim (us - su), \qquad \Delta_{ds} \sim (ds - sd)$

three-flavor systems

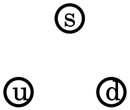
- antisymmetric flavor combinations:

  $\Delta_{ud} \sim (ud - du), \qquad \Delta_{us} \sim (us - su), \qquad \Delta_{ds} \sim (ds - sd)$

- 8 different phases:

  normal quark matter (NQ)

  $\Delta_{ud} = \Delta_{us} = \Delta_{ds} = 0$



three-flavor systems

TECHNISCHE UNIVERSITÄT DARMSTADT

- antisymmetric flavor combinations:

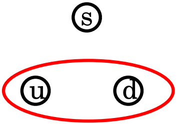
  $\Delta_{ud} \sim (ud - du), \qquad \Delta_{us} \sim (us - su), \qquad \Delta_{ds} \sim (ds - sd)$

- 8 different phases:

  2SC phase

  $\Delta_{ud} \neq 0, \quad \Delta_{us} = \Delta_{ds} = 0$

  + two more phases of this kind



three-flavor systems

- antisymmetric flavor combinations:

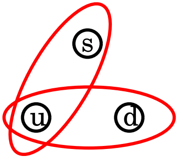
$\Delta_{ud} \sim (ud - du), \qquad \Delta_{us} \sim (us - su), \qquad \Delta_{ds} \sim (ds - sd)$

- 8 different phases:

uSC phase

$\Delta_{ud},\ \Delta_{us} \neq 0, \qquad \Delta_{ds} = 0$

+ two more phases of this kind



three-flavor systems

TECHNISCHE UNIVERSITÄT DARMSTADT

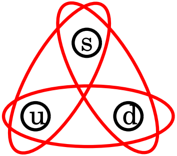
- antisymmetric flavor combinations:

  $\Delta_{ud} \sim (ud - du), \qquad \Delta_{us} \sim (us - su), \qquad \Delta_{ds} \sim (ds - sd)$

- 8 different phases:

  CFL phase

  $\Delta_{ud},\ \Delta_{us},\ \Delta_{ds} \neq 0$



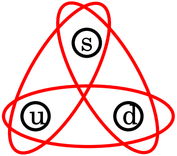
- antisymmetric flavor combinations:

  $\Delta_{ud} \sim (ud - du), \qquad \Delta_{us} \sim (us - su), \qquad \Delta_{ds} \sim (ds - sd)$

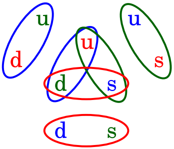
- 8 different phases:

  CFL phase

  $\Delta_{ud}, \; \Delta_{us}, \; \Delta_{ds} \neq 0$



- additional color structure:



Which phase is favored?

TECHNISCHE UNIVERSITÄT DARMSTADT

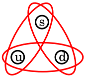
- precondition for BCS pairing: $\delta p_F \lesssim \frac{\Delta}{\sqrt{2}}$, $\quad p_F^i = \sqrt{\mu_i^2 - M_i^2}$

Which phase is favored?

TECHNISCHE UNIVERSITÄT DARMSTADT

- precondition for BCS pairing: $\delta p_F \lesssim \frac{\Delta}{\sqrt{2}}$, $\quad p_F^i = \sqrt{\mu_i^2 - M_i^2}$
- high densities: $\mu \gg M_s$

  ➜ $p_F^s \approx p_F^d \approx p_F^u$ ➜ CFL

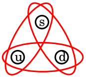


Which phase is favored?

TECHNISCHE UNIVERSITÄT DARMSTADT

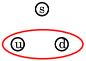
- precondition for BCS pairing: $\delta p_F \lesssim \frac{\Delta}{\sqrt{2}}$, $\quad p_F^i = \sqrt{\mu_i^2 - M_i^2}$
- high densities: $\mu \gg M_s$

  ➜ $p_F^s \approx p_F^d \approx p_F^u$ ➜ CFL



- moderate densities: $M_s \gtrsim \mu \gg M_{u,d}$

  ➜ $p_F^s \ll p_F^d \approx p_F^u$ ➜ 2SC

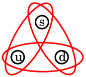


Which phase is favored?

TECHNISCHE UNIVERSITÄT DARMSTADT

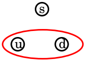
- precondition for BCS pairing: $\delta p_F \lesssim \frac{\Delta}{\sqrt{2}}, \qquad p_F^i = \sqrt{\mu_i^2 - M_i^2}$
- high densities: $\mu \gg M_s$

  ➜ $p_F^s \approx p_F^d \approx p_F^u$ ➜ CFL



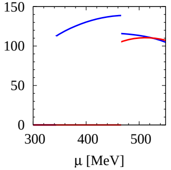
- moderate densities: $M_s \gtrsim \mu \gg M_{u,d}$

  ➜ $p_F^s \ll p_F^d \approx p_F^u$ ➜ 2SC

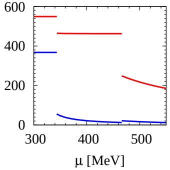


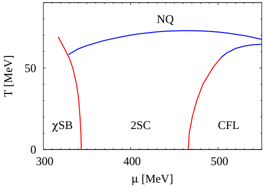
- NJL-model results: [M.B.; M. Oertel, NPA (2002); M. Oertel, M.B., hep-ph/0202098]

Δ_{ud}, $\Delta_{us} = \Delta_{us}$



$M_u = M_d$, M_s





Neutral quark matter

- constraints in compact stars:
  - color neutrality: $n_r = n_g = n_b$
  - electric neutrality: $n_Q = \frac{2}{3} n_u - \frac{1}{3} n_d - \frac{1}{3} n_s - n_e = 0$
  - β equilibrium: $\mu_e = \mu_d - \mu_u$ ➜ $n_e \ll n_{u,d}$

Neutral quark matter

TECHNISCHE UNIVERSITÄT DARMSTADT

- constraints in compact stars:
  - color neutrality: *(minor effect)*
  - electric neutrality:
  - β equilibrium:

  (electric neutrality and β equilibrium:) $\frac{2}{3}n_u - \frac{1}{3}n_d - \frac{1}{3}n_s \approx 0$

Neutral quark matter

- constraints in compact stars:
  - color neutrality: *(minor effect)*
  - electric neutrality:
  - β equilibrium:

  $$\left.\begin{array}{l}\text{electric neutrality} \\ \beta \text{ equilibrium}\end{array}\right\} \quad \frac{2}{3}n_u - \frac{1}{3}n_d - \frac{1}{3}n_s \approx 0$$

➜ all flavors have different Fermi momenta

Neutral quark matter

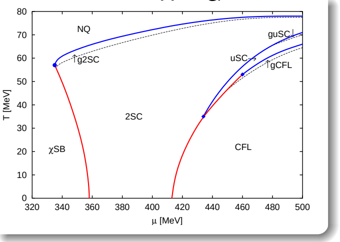
TECHNISCHE UNIVERSITÄT DARMSTADT

- constraints in compact stars:
  - color neutrality: *(minor effect)*
  - electric neutrality:
  - β equilibrium:

  $$\left.\begin{array}{l}\text{electric neutrality}\\ \beta \text{ equilibrium}\end{array}\right\} \quad \tfrac{2}{3}n_u - \tfrac{1}{3}n_d - \tfrac{1}{3}n_s \approx 0$$

  ➜ all flavors have different Fermi momenta
- phase diagram: [Rüster, Werth, M.B., Shovkovy, Rischke, PRD (2005)]

$H = G$



Neutral quark matter

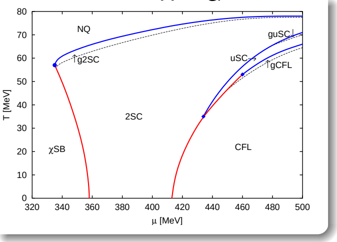
- constraints in compact stars:
  - color neutrality: *(minor effect)*
  - electric neutrality:
  - β equilibrium:

$$\left.\begin{array}{l} \\ \\ \end{array}\right\} \quad \tfrac{2}{3} n_u - \tfrac{1}{3} n_d - \tfrac{1}{3} n_s \approx 0$$

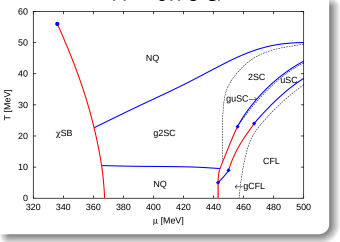
➜ all flavors have different Fermi momenta

- phase diagram: [Rüster, Werth, M.B., Shovkovy, Rischke, PRD (2005)]

$H = G$



$H = 0.75\,G$



Kaon condensation in the CFL phase

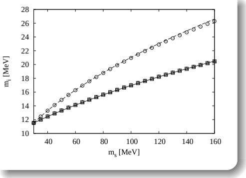
- CFL: chiral symmetry broken
  - ➜ Goldstone bosons
    - small masses: $\sim \mathcal{O}(10\text{ MeV})$

      [Son, Stephanov, PRD (2000)]
    - $m_s > m_{u,d}$ ➜ $\mu_s^{eff} \simeq \frac{m_s^2 - m_u^2}{2\mu}$
  - ➜ K^0 condensation

    [T. Schäfer, PRL (2000); Bedaque, Schäfer, NPA (2002)]

- NJL model:

  [V. Kleinhaus, M.B., D. Nickel, M. Oertel, PRD (2007)]



Kaon condensation in the CFL phase

TECHNISCHE UNIVERSITÄT DARMSTADT

- CFL: chiral symmetry broken
  - ➜ Goldstone bosons
    - small masses: $\sim \mathcal{O}(10\text{ MeV})$

      [Son, Stephanov, PRD (2000)]
    - $m_s > m_{u,d}$ ➜ $\mu_s^{eff} \simeq \frac{m_s^2 - m_u^2}{2\mu}$
  - ➜ K^0 condensation

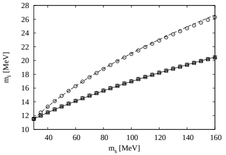
    [T. Schäfer, PRL (2000); Bedaque, Schäfer, NPA (2002)]

- phase diagram:
  - include pseudoscalar diquark condesates

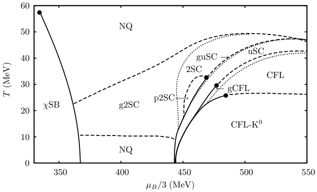
    [M.B., PLB (2005); M.M. Forbes, PDR (2005)]

- NJL model:

[V. Kleinhaus, M.B., D. Nickel, M. Oertel, PRD (2007)]



[H. Warringa, hep-ph/0606063]



Kaon condensation in the CFL phase

TECHNISCHE UNIVERSITÄT DARMSTADT

- CFL: chiral symmetry broken
  - ➜ Goldstone bosons
    - small masses: $\sim \mathcal{O}(10 \text{ MeV})$

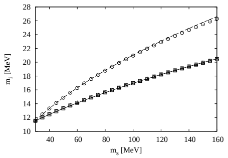
      [Son, Stephanov, PRD (2000)]
    - $m_s > m_{u,d}$ ➜ $\mu_s^{eff} \simeq \frac{m_s^2 - m_u^2}{2\mu}$
  - ➜ K^0 condensation

    [T. Schäfer, PRL (2000); Bedaque, Schäfer, NPA (2002)]
- phase diagram:
  - include pseudoscalar diquark condesates

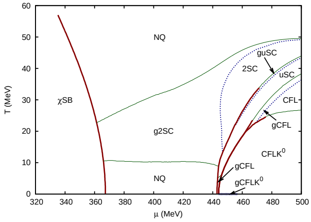
    [M.B., PLB (2005); M.M. Forbes, PDR (2005)]
- see talk by Hannes Basler!

- NJL model:

[V. Kleinhaus, M.B., D. Nickel, M. Oertel, PRD (2007)]



[H. Basler, M.B., arXiv:0912.3411]



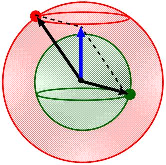
INHOMOGENEOUS PHASES

LOFF phases

- BCS pairing disfavored for $\delta p_F \gtrsim \frac{\Delta}{\sqrt{2}}$
- alternative: pairs with nonzero total momentum

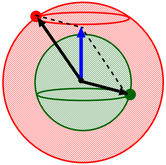
LOFF phases

- BCS pairing disfavored for $\delta p_F \gtrsim \frac{\Delta}{\sqrt{2}}$
- alternative: pairs with nonzero total momentum
- FF: [P. Fulde, R.A. Ferrell, Phys. Rev., 1964]
  - single plane wave, $\langle q(\vec{x})q(\vec{x})\rangle \sim \Delta\, e^{2i\vec{q}\cdot\vec{x}}$ for fixed $\vec{q}$
  - disfavored by phase space



LOFF phases

- BCS pairing disfavored for $\delta p_F \gtrsim \frac{\Delta}{\sqrt{2}}$
- alternative: pairs with nonzero total momentum
- FF: [P. Fulde, R.A. Ferrell, Phys. Rev., 1964]
  - single plane wave, $\langle q(\vec{x})q(\vec{x})\rangle \sim \Delta\, e^{2i\vec{q}\cdot\vec{x}}$ for fixed $\vec{q}$
  - disfavored by phase space

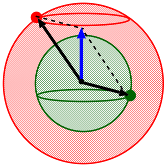


- LO: [Larkin, Ovchinnikov, Zh. Eksp. Teor. Fiz., 1964]
  - multiple plane waves (e.g., $\cos(2\vec{q}\cdot\vec{x})$)

LOFF phases

TECHNISCHE UNIVERSITÄT DARMSTADT

- BCS pairing disfavored for $\delta p_F \gtrsim \frac{\Delta}{\sqrt{2}}$
- alternative: pairs with nonzero total momentum



- FF: [P. Fulde, R.A. Ferrell, Phys. Rev., 1964]
  - single plane wave, $\langle q(\vec{x})q(\vec{x})\rangle \sim \Delta\, e^{2i\vec{q}\cdot\vec{x}}$ for fixed $\vec{q}$
  - disfavored by phase space
- LO: [Larkin, Ovchinnikov, Zh. Eksp. Teor. Fiz., 1964]
  - multiple plane waves (e.g., $\cos(2\vec{q}\cdot\vec{x})$)
- NJL-model: mostly FF ansatz

[M. Alford, J. Bowers, K. Rajagopal, PRD (2001); A. Sedrakian, D.H. Rischke, PRD (2009); . . .]

NJL-model for LO phases

[D. Nickel, M.B., PRD (2009)]

TECHNISCHE UNIVERSITÄT DARMSTADT

- generalized mean-fields: $\langle q^T(x)\, C\gamma_5\, \tau_2\, \lambda_2\, q(x)\rangle = -2H\Delta(x)$
  - $\Delta(x)$ classical fields
  - retain space dependence!
  - periodic ansatz: $\Delta(x) = \sum_{\vec{q}_k \in R.L.} \Delta_{\vec{q}_k} \exp(i\vec{q}_k \cdot \vec{x})$, $(R.L. =$ reciprocal lattice)

NJL-model for LO phases

[D. Nickel, M.B., PRD (2009)]

TECHNISCHE UNIVERSITÄT DARMSTADT

- generalized mean-fields: $\langle q^T(x)\, C\gamma_5\, \tau_2\, \lambda_2\, q(x)\rangle = -2H\Delta(x)$
  - $\Delta(x)$ classical fields
  - retain space dependence!
  - periodic ansatz: $\Delta(x) = \sum_{\vec{q}_k \in R.L.} \Delta_{\vec{q}_k} \exp(i\vec{q}_k \cdot \vec{x})\,,$ $\quad (R.L. =$ reciprocal lattice$)$
- mean-field Lagrangian: $\mathcal{L}_{MF} + \mu_i\, q_i^\dagger q_i \;=\; \bar{\Psi}(x)\, S^{-1}(x)\, \Psi(x) \;-\; \frac{|\Delta(x)|^2}{4H}$

NJL-model for LO phases

[D. Nickel, M.B., PRD (2009)]

TECHNISCHE UNIVERSITÄT DARMSTADT

- generalized mean-fields: $\langle q^T(x)\, C\gamma_5\, \tau_2\, \lambda_2\, q(x)\rangle = -2H\Delta(x)$
  - $\Delta(x)$ classical fields
  - retain space dependence!
  - periodic ansatz: $\Delta(x) = \sum_{\vec{q}_k \in R.L.} \Delta_{\vec{q}_k} \exp(i\vec{q}_k \cdot \vec{x})\,,$ ($R.L.$ = reciprocal lattice)
- mean-field Lagrangian: $\mathcal{L}_{MF} + \mu_i\, q_i^\dagger q_i = \bar{\Psi}(x)\, S^{-1}(x)\, \Psi(x) - \frac{|\Delta(x)|^2}{4H}$
  - still bilinear ➜ $\Omega_{MF}(T,\mu) = -\frac{1}{2}\frac{T}{V}\mathrm{Tr}\ln\frac{S^{-1}}{T} + \sum_{\vec{q}_k}\frac{|\Delta_{\vec{q}_k}|^2}{4H}$

NJL-model for LO phases

[D. Nickel, M.B., PRD (2009)]

TECHNISCHE UNIVERSITÄT DARMSTADT

- generalized mean-fields: $\langle q^T(x)\, C\gamma_5\, \tau_2\, \lambda_2\, q(x)\rangle = -2H\Delta(x)$
  - $\Delta(x)$ classical fields
  - retain space dependence!
  - periodic ansatz: $\Delta(x) = \sum_{\vec{q}_k \in R.L.} \Delta_{\vec{q}_k} \exp(i\vec{q}_k \cdot \vec{x})$, ($R.L.$ = reciprocal lattice)
- mean-field Lagrangian: $\mathcal{L}_{MF} + \mu_i\, q_i^\dagger q_i = \bar{\Psi}(x)\, S^{-1}(x)\, \Psi(x) - \frac{|\Delta(x)|^2}{4H}$
  - still bilinear ➜ $\Omega_{MF}(T,\mu) = -\frac{1}{2}\frac{T}{V}\mathrm{Tr}\ln\frac{S^{-1}}{T} + \sum_{\vec{q}_k}\frac{|\Delta_{\vec{q}_k}|^2}{4H}$
  - but $\mathrm{Tr}\ln S^{-1}$ nontrivial because the propagator is more complicated:

$$S^{-1}_{\vec{p}_m,\vec{p}_n} = \begin{pmatrix} (\not{p}_n + \hat{\mu}\gamma^0)\,\delta_{\vec{p}_m,\vec{p}_n} & \sum_{\vec{q}_k} \Delta_{q_k}\delta_{\vec{q}_k,\vec{p}_m-\vec{p}_n}\gamma_5\tau_2\lambda_2 \\ -\sum_{\vec{q}_k} \Delta^*_{q_k}\delta_{\vec{q}_k,\vec{p}_n-\vec{p}_m}\gamma_5\tau_2\lambda_2 & (\not{p}_n - \hat{\mu}\gamma^0)\,\delta_{\vec{p}_m,\vec{p}_n} \end{pmatrix}$$

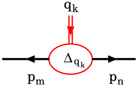
NJL-model for LO phases

[D. Nickel, M.B., PRD (2009)]

- generalized mean-fields: $\langle q^T(x)\, C\gamma_5\, \tau_2\, \lambda_2\, q(x)\rangle = -2H\Delta(x)$
  - $\Delta(x)$ classical fields
  - retain space dependence!
  - periodic ansatz: $\Delta(x) = \sum_{\vec{q}_k \in R.L.} \Delta_{\vec{q}_k} \exp(i\vec{q}_k \cdot \vec{x})\,,$ $\quad (R.L. = \text{reciprocal lattice})$
- mean-field Lagrangian: $\mathcal{L}_{MF} + \mu_i\, q_i^\dagger q_i \;=\; \bar{\Psi}(x)\, S^{-1}(x)\, \Psi(x) - \frac{|\Delta(x)|^2}{4H}$
  - still bilinear ➜ $\Omega_{MF}(T,\mu) = -\frac{1}{2}\frac{T}{V}\mathrm{Tr}\ln\frac{S^{-1}}{T} + \sum_{\vec{q}_k}\frac{|\Delta_{\vec{q}_k}|^2}{4H}$
  - but $\mathrm{Tr}\ln S^{-1}$ nontrivial because the propagator is more complicated:

$$S^{-1}_{\vec{p}_m,\vec{p}_n} = \begin{pmatrix} (\not{p}_n + \hat{\mu}\gamma^0)\,\delta_{\vec{p}_m,\vec{p}_n} & \sum_{\vec{q}_k} \Delta_{q_k}\delta_{\vec{q}_k,\vec{p}_m-\vec{p}_n}\gamma_5\tau_2\lambda_2 \\ -\sum_{\vec{q}_k} \Delta^*_{q_k}\delta_{\vec{q}_k,\vec{p}_n-\vec{p}_m}\gamma_5\tau_2\lambda_2 & (\not{p}_n - \hat{\mu}\gamma^0)\,\delta_{\vec{p}_m,\vec{p}_n} \end{pmatrix}$$

  - The condensates couple different momenta!



Results

[D. Nickel, M.B., PRD (2009)]

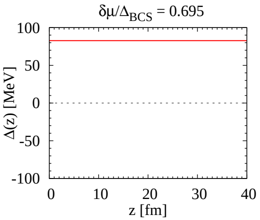
- simplifications:
  - two fermion species with $\mu_i = \bar{\mu} \pm \delta\mu$
  - real gap functions with 1-dim modulations: $\Delta(z) = \sum_k \Delta_k \, e^{2ikqz}, \quad \Delta_k = \Delta^*_{-k}$

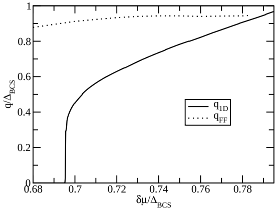
Results

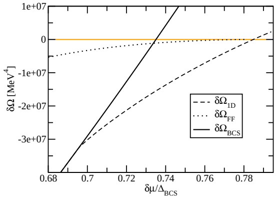
[D. Nickel, M.B., PRD (2009)]

TECHNISCHE UNIVERSITÄT DARMSTADT

- simplifications:
  - two fermion species with $\mu_i = \bar{\mu} \pm \delta\mu$
  - real gap functions with 1-dim modulations: $\Delta(z) = \sum_k \Delta_k \, e^{2ikqz}, \quad \Delta_k = \Delta^*_{-k}$
- favored gap functions: preferred q: free-energy gain:





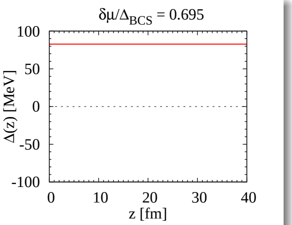


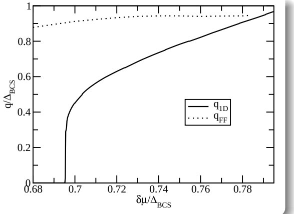
Results

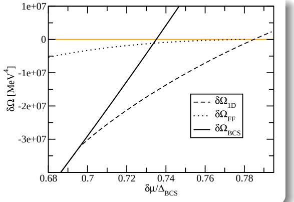
[D. Nickel, M.B., PRD (2009)]

TECHNISCHE UNIVERSITÄT DARMSTADT

- simplifications:
  - two fermion species with $\mu_i = \bar{\mu} \pm \delta\mu$
  - real gap functions with 1-dim modulations: $\Delta(z) = \sum_k \Delta_k \, e^{2ikqz}, \quad \Delta_k = \Delta^*_{-k}$
- favored gap functions: preferred q: free-energy gain:







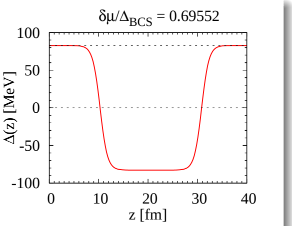
- increasing $\delta\mu$: BCS

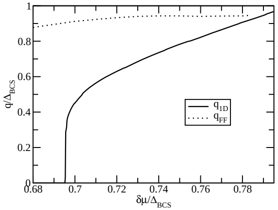
Results

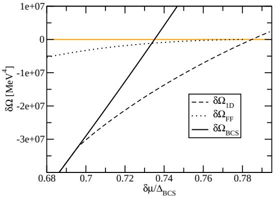
[D. Nickel, M.B., PRD (2009)]

TECHNISCHE UNIVERSITÄT DARMSTADT

- simplifications:
  - two fermion species with $\mu_i = \bar{\mu} \pm \delta\mu$
  - real gap functions with 1-dim modulations: $\Delta(z) = \sum_k \Delta_k \, e^{2ikqz}, \quad \Delta_k = \Delta^*_{-k}$
- favored gap functions: preferred q: free-energy gain:







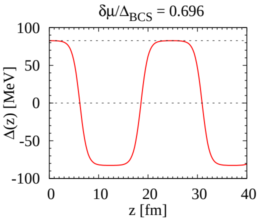
- increasing $\delta\mu$: BCS $\rightarrow$ soliton lattice

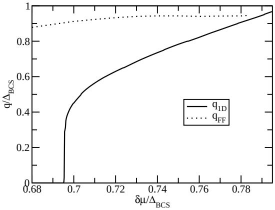
Results

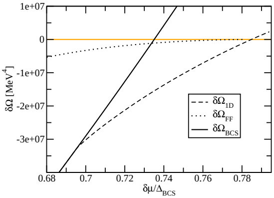
[D. Nickel, M.B., PRD (2009)]

TECHNISCHE UNIVERSITÄT DARMSTADT

- simplifications:
  - two fermion species with $\mu_i = \bar{\mu} \pm \delta\mu$
  - real gap functions with 1-dim modulations: $\Delta(z) = \sum_k \Delta_k \, e^{2ikqz}, \quad \Delta_k = \Delta^*_{-k}$
- favored gap functions: preferred q: free-energy gain:







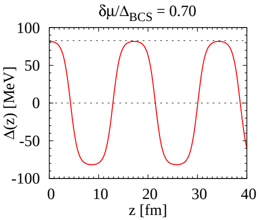
- increasing $\delta\mu$: BCS $\rightarrow$ soliton lattice

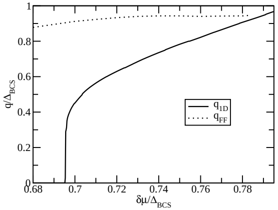
Results

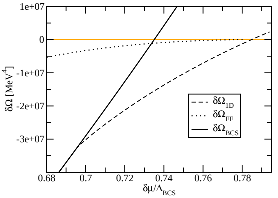
[D. Nickel, M.B., PRD (2009)]

TECHNISCHE UNIVERSITÄT DARMSTADT

- simplifications:
  - two fermion species with $\mu_i = \bar{\mu} \pm \delta\mu$
  - real gap functions with 1-dim modulations: $\Delta(z) = \sum_k \Delta_k \, e^{2ikqz}, \quad \Delta_k = \Delta^*_{-k}$
- favored gap functions: preferred q: free-energy gain:





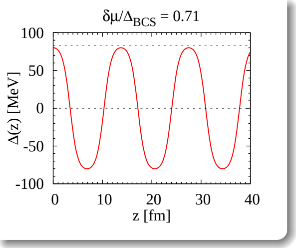


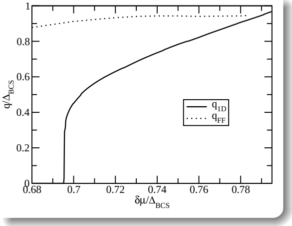
- increasing $\delta\mu$: BCS $\rightarrow$ soliton lattice

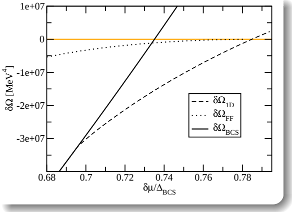
Results

[D. Nickel, M.B., PRD (2009)]

- simplifications:
  - two fermion species with $\mu_i = \bar{\mu} \pm \delta\mu$
  - real gap functions with 1-dim modulations: $\Delta(z) = \sum_k \Delta_k \, e^{2ikqz}, \quad \Delta_k = \Delta^*_{-k}$
- favored gap functions: preferred q: free-energy gain:





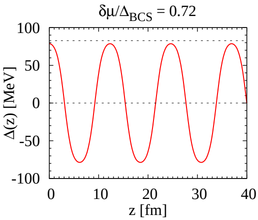


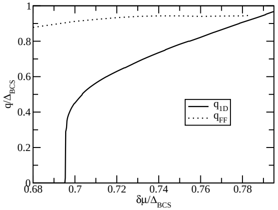
- increasing $\delta\mu$: BCS $\rightarrow$ soliton lattice

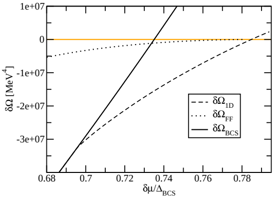
Results

[D. Nickel, M.B., PRD (2009)]

- simplifications:
  - two fermion species with $\mu_i = \bar{\mu} \pm \delta\mu$
  - real gap functions with 1-dim modulations: $\Delta(z) = \sum_k \Delta_k \, e^{2ikqz}, \quad \Delta_k = \Delta^*_{-k}$
- favored gap functions: preferred q: free-energy gain:







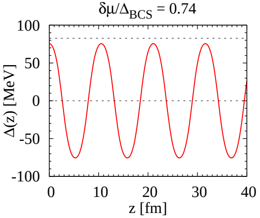
- increasing $\delta\mu$: BCS $\rightarrow$ (soliton lattice $\rightarrow$ sinusoidal)

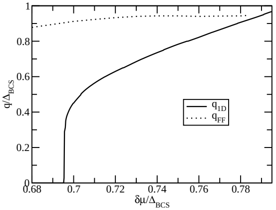
Results

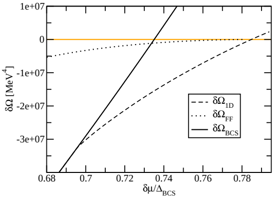
[D. Nickel, M.B., PRD (2009)]

TECHNISCHE UNIVERSITÄT DARMSTADT

- simplifications:
  - two fermion species with $\mu_i = \bar{\mu} \pm \delta\mu$
  - real gap functions with 1-dim modulations: $\Delta(z) = \sum_k \Delta_k\, e^{2ikqz}, \quad \Delta_k = \Delta^*_{-k}$
- favored gap functions: preferred q: free-energy gain:







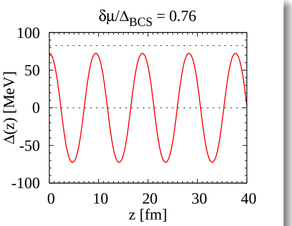
- increasing $\delta\mu$: BCS $\rightarrow$ (soliton lattice $\rightarrow$ sinusoidal)

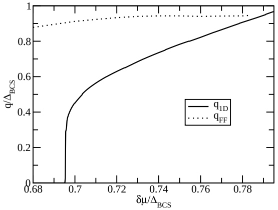
Results

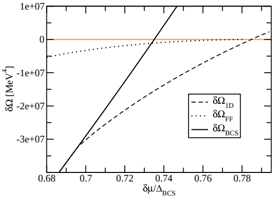
[D. Nickel, M.B., PRD (2009)]

TECHNISCHE UNIVERSITÄT DARMSTADT

- simplifications:
  - two fermion species with $\mu_i = \bar{\mu} \pm \delta\mu$
  - real gap functions with 1-dim modulations: $\Delta(z) = \sum_k \Delta_k \, e^{2ikqz}, \quad \Delta_k = \Delta^*_{-k}$
- favored gap functions: preferred q: free-energy gain:







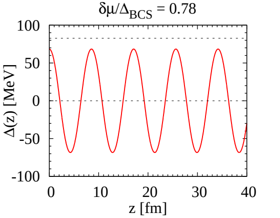
- increasing $\delta\mu$: BCS $\rightarrow$ (soliton lattice $\rightarrow$ sinusoidal)

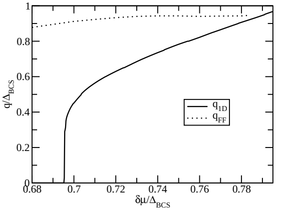
Results

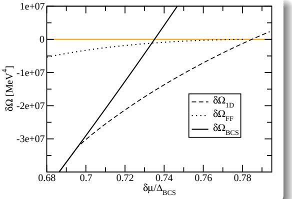
[D. Nickel, M.B., PRD (2009)]

TECHNISCHE UNIVERSITÄT DARMSTADT

- simplifications:
  - two fermion species with $\mu_i = \bar{\mu} \pm \delta\mu$
  - real gap functions with 1-dim modulations: $\Delta(z) = \sum_k \Delta_k \, e^{2ikqz}, \quad \Delta_k = \Delta^*_{-k}$
- favored gap functions: preferred q: free-energy gain:







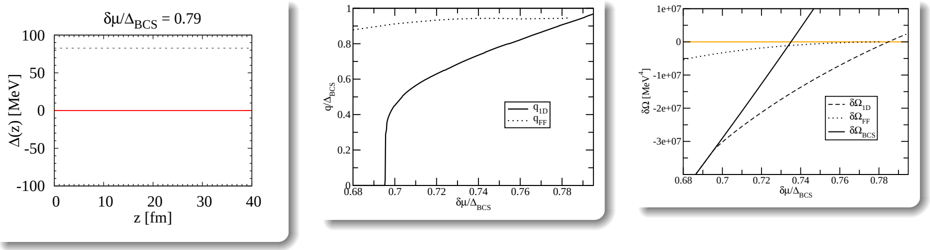
- increasing $\delta\mu$: BCS $\rightarrow$ (soliton lattice $\rightarrow$ sinusoidal)

Results

[D. Nickel, M.B., PRD (2009)]

TECHNISCHE UNIVERSITÄT DARMSTADT

- simplifications:
  - two fermion species with $\mu_i = \bar{\mu} \pm \delta\mu$
  - real gap functions with 1-dim modulations: $\Delta(z) = \sum_k \Delta_k\, e^{2ikqz}, \quad \Delta_k = \Delta^*_{-k}$
- favored gap functions: preferred q: free-energy gain:

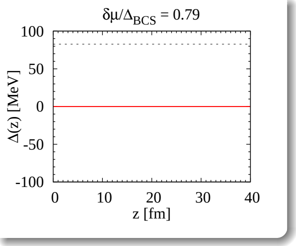


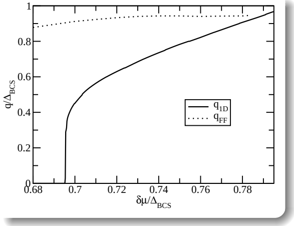
- increasing $\delta\mu$: BCS $\rightarrow$ (soliton lattice $\rightarrow$ sinusoidal) $\rightarrow$ normal phase

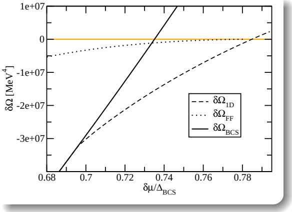
Results

[D. Nickel, M.B., PRD (2009)]

- simplifications:
  - two fermion species with $\mu_i = \bar{\mu} \pm \delta\mu$
  - real gap functions with 1-dim modulations: $\Delta(z) = \sum_k \Delta_k \, e^{2ikqz}, \quad \Delta_k = \Delta^*_{-k}$
- favored gap functions: preferred q: free-energy gain:







- increasing $\delta\mu$: BCS $\rightarrow$ (soliton lattice $\rightarrow$ sinusoidal) $\rightarrow$ normal phase
- LO solution much more favored than FF

Inhomogeneous chiral condensates

TECHNISCHE UNIVERSITÄT DARMSTADT

- chiral density wave:

  [Broniowski, Kotlorz, Kutschera, Acta Phys. Pol. (1991); Nakano, Tatsumi PRD (2005), ...]

  - $\langle \bar{q}q \rangle = \phi_0 \cos(qz), \quad \langle \bar{q}\, i\gamma_5 \tau_3 q \rangle = \phi_0 \, sin(qz)$

  ➜ $M(z) = m - 2G\,\phi_0\, e^{iqz\gamma_5}$ "chiral spiral"

Inhomogeneous chiral condensates

TECHNISCHE UNIVERSITÄT DARMSTADT

- chiral density wave:

  [Broniowski, Kotlorz, Kutschera, Acta Phys. Pol. (1991); Nakano, Tatsumi PRD (2005), ...]

  - $\langle \bar{q}q \rangle = \phi_0 \cos(qz), \quad \langle \bar{q}\, i\gamma_5\tau_3 q \rangle = \phi_0 \sin(qz)$

  ➜ $M(z) = m - 2G\,\phi_0\, e^{iqz\gamma_5}$ "chiral spiral"

- general 1-dim modulations:

  - analytically known for the 1+1 D (chiral) Gross-Neveu model

    [Schnetz, Thies, Urlichs, Annals. Phys. (2004,2006)]

    chiral limit: $M_{1+1}(z) = \sqrt{\nu}\kappa\, \mathrm{sn}\big(\kappa(z - z_0); \nu\big)$, sn = Jacobi elliptic functions

Inhomogeneous chiral condensates

TECHNISCHE UNIVERSITÄT DARMSTADT

- chiral density wave:

  [Broniowski, Kotlorz, Kutschera, Acta Phys. Pol. (1991); Nakano, Tatsumi PRD (2005), ...]

  - $\langle \bar{q}q \rangle = \phi_0 \cos(qz)$, $\langle \bar{q}\, i\gamma_5\tau_3 q \rangle = \phi_0\, sin(qz)$

  ➜ $M(z) = m - 2G\,\phi_0\, e^{iqz\gamma_5}$ "chiral spiral"

- general 1-dim modulations:
  - analytically known for the 1+1 D (chiral) Gross-Neveu model

    [Schnetz, Thies, Urlichs, Annals. Phys. (2004,2006)]

    chiral limit: $M_{1+1}(z) = \sqrt{\nu}\kappa\, \mathrm{sn}\big(\kappa(z - z_0); \nu\big)$, sn = Jacobi elliptic functions

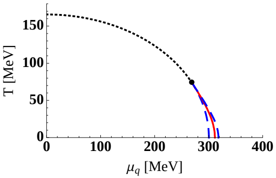
  - can be lifted to 1-dim modulations in 3+1 D: [D. Nickel, PRD (2009)]

  ➜ minimize Ω w.r.t. κ and ν

Results

[D. Nickel, PRL (2009), PRD (2009); S. Carignano, D. Nickel, M.B., in prep.]

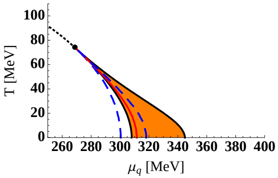
phase diagram



Results

[D. Nickel, PRL (2009), PRD (2009); S. Carignano, D. Nickel, M.B., in prep.]

phase diagram

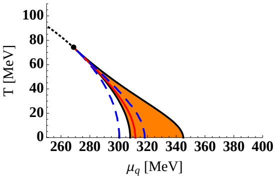


Results

[D. Nickel, PRL (2009), PRD (2009); S. Carignano, D. Nickel, M.B., in prep.]

TECHNISCHE UNIVERSITÄT DARMSTADT

phase diagram



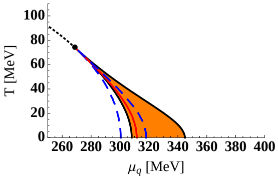
- original 1st-order line completely covered by the inhomogeneous phase!

Results

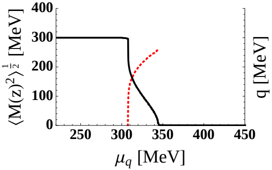
[D. Nickel, PRL (2009), PRD (2009); S. Carignano, D. Nickel, M.B., in prep.]

TECHNISCHE UNIVERSITÄT DARMSTADT

phase diagram



$\sqrt{\langle M^2(z)\rangle}$, wave number

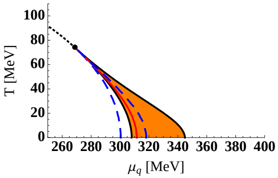


- original 1st-order line completely covered by the inhomogeneous phase!

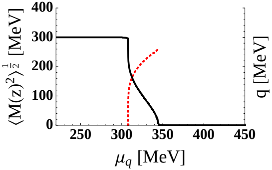
Results

[D. Nickel, PRL (2009), PRD (2009); S. Carignano, D. Nickel, M.B., in prep.]

phase diagram



$\sqrt{\langle M^2(z) \rangle}$, wave number



- original 1st-order line completely covered by the inhomogeneous phase!
- inhom. $\longrightarrow$ hom. broken: $q \longrightarrow 0$

  inhom. $\longrightarrow$ restored: $\langle M^2 \rangle \longrightarrow 0$

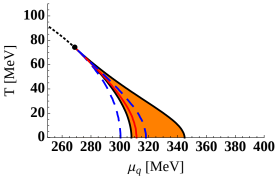
  ➜ both phase boundaries 2nd order!

Results

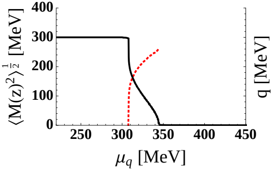
[D. Nickel, PRL (2009), PRD (2009); S. Carignano, D. Nickel, M.B., in prep.]

TECHNISCHE UNIVERSITÄT DARMSTADT

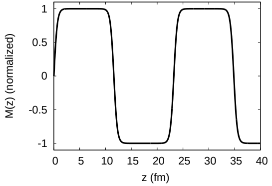
phase diagram



$\sqrt{\langle M^2(z)\rangle}$, wave number



$M(z)$ (μ = 307 MeV)

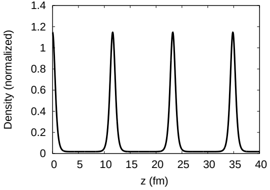


- original 1st-order line completely covered by the inhomogeneous phase!
- inhom. $\longrightarrow$ hom. broken: $q \longrightarrow 0$

  inhom. $\longrightarrow$ restored: $\langle M^2\rangle \longrightarrow 0$

  ➜ both phase boundaries 2nd order!

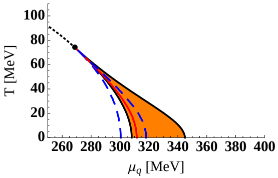
density (μ = 307 MeV)



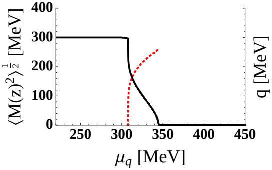
Results

[D. Nickel, PRL (2009), PRD (2009); S. Carignano, D. Nickel, M.B., in prep.]

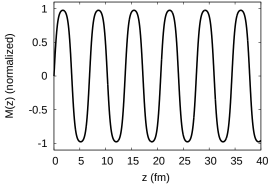
phase diagram



$\sqrt{\langle M^2(z)\rangle}$, wave number



$M(z)$ (μ = 308 MeV)

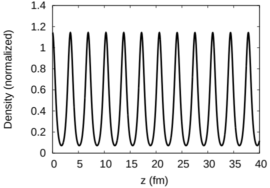


- original 1st-order line completely covered by the inhomogeneous phase!
- inhom. $\rightarrow$ hom. broken: $q \rightarrow 0$

  inhom. $\rightarrow$ restored: $\langle M^2 \rangle \rightarrow 0$

  ➜ both phase boundaries 2nd order!

density (μ = 308 MeV)

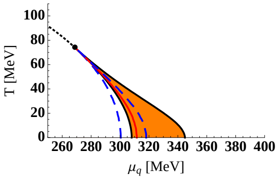


Results

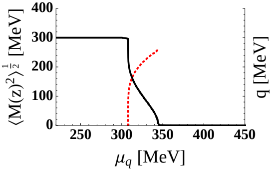
[D. Nickel, PRL (2009), PRD (2009); S. Carignano, D. Nickel, M.B., in prep.]

TECHNISCHE UNIVERSITÄT DARMSTADT

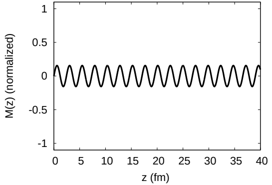
phase diagram



$\sqrt{\langle M^2(z)\rangle}$, wave number



$M(z)$ (μ = 342 MeV)

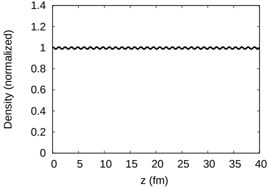


- original 1st-order line completely covered by the inhomogeneous phase!
- inhom. $\rightarrow$ hom. broken: $q \rightarrow 0$

  inhom. $\rightarrow$ restored: $\langle M^2 \rangle \rightarrow 0$

  ➜ both phase boundaries 2nd order!

density (μ = 342 MeV)

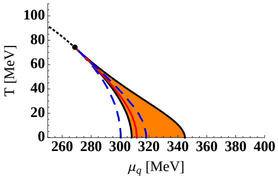


Results

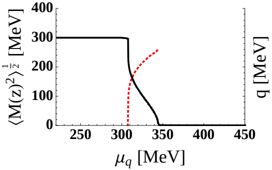
TECHNISCHE UNIVERSITÄT DARMSTADT

[D. Nickel, PRL (2009), PRD (2009); S. Carignano, D. Nickel, M.B., in prep.]

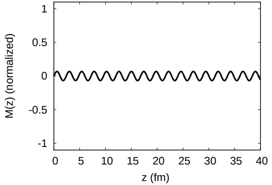
phase diagram



$\sqrt{\langle M^2(z)\rangle}$, wave number



$M(z)$ (μ = 344 MeV)

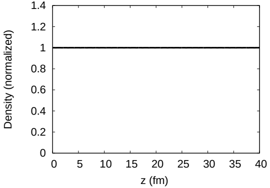


- original 1st-order line completely covered by the inhomogeneous phase!
- inhom. $\longrightarrow$ hom. broken: $q \longrightarrow 0$

  inhom. $\longrightarrow$ restored: $\langle M^2\rangle \longrightarrow 0$

  ➜ both phase boundaries 2nd order!

density (μ = 344 MeV)



Influence of vector interactions

TECHNISCHE UNIVERSITÄT DARMSTADT

- include vector interaction: $\mathcal{L}_V = -G_V(\bar{q}\gamma^\mu q)^2$

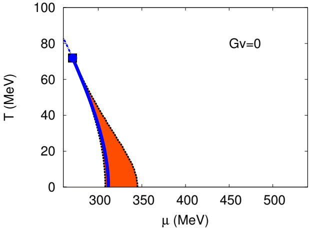
Influence of vector interactions

TECHNISCHE UNIVERSITÄT DARMSTADT

- include vector interaction: $\mathcal{L}_V = -G_V(\bar{q}\gamma^\mu q)^2$
- approximation: $\tilde{\mu}(z) = \mu - 2G_V\, n(z) \;\to\; \mu - 2G_V \langle n(z) \rangle = const.$

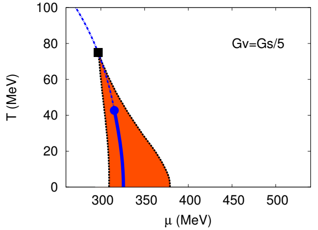
Influence of vector interactions

- include vector interaction: $\mathcal{L}_V = -G_V(\bar{q}\gamma^\mu q)^2$
- approximation: $\tilde{\mu}(z) = \mu - 2G_V\, n(z) \;\rightarrow\; \mu - 2G_V \left\langle n(z) \right\rangle = const.$
- phase diagram: [S. Carignano, D. Nickel, M.B., in prep.]



Influence of vector interactions

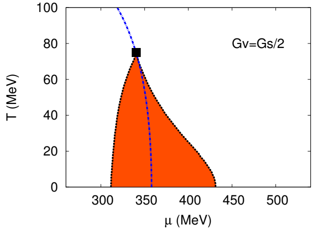
- include vector interaction: $\mathcal{L}_V = -G_V(\bar{q}\gamma^\mu q)^2$
- approximation: $\tilde{\mu}(z) = \mu - 2G_V\, n(z) \;\rightarrow\; \mu - 2G_V \langle n(z) \rangle = const.$
- phase diagram: [S. Carignano, D. Nickel, M.B., in prep.]



Influence of vector interactions

TECHNISCHE UNIVERSITÄT DARMSTADT

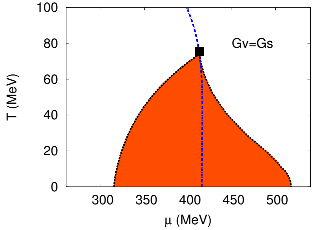
- include vector interaction: $\mathcal{L}_V = -G_V(\bar{q}\gamma^\mu q)^2$
- approximation: $\tilde{\mu}(z) = \mu - 2G_V\, n(z) \;\rightarrow\; \mu - 2G_V \langle n(z) \rangle = const.$
- phase diagram: [S. Carignano, D. Nickel, M.B., in prep.]



Influence of vector interactions

TECHNISCHE UNIVERSITÄT DARMSTADT

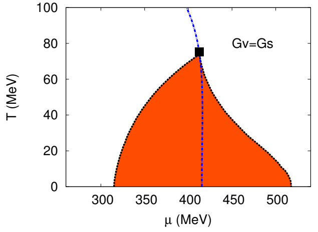
- include vector interaction: $\mathcal{L}_V = -G_V(\bar{q}\gamma^\mu q)^2$
- approximation: $\tilde{\mu}(z) = \mu - 2G_V\, n(z) \;\rightarrow\; \mu - 2G_V \langle n(z) \rangle = const.$
- phase diagram: [S. Carignano, D. Nickel, M.B., in prep.]



Influence of vector interactions

TECHNISCHE UNIVERSITÄT DARMSTADT

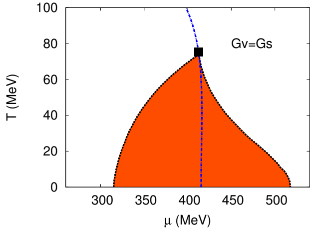
- include vector interaction: $\mathcal{L}_V = -G_V(\bar{q}\gamma^\mu q)^2$
- approximation: $\tilde{\mu}(z) = \mu - 2G_V\, n(z) \;\rightarrow\; \mu - 2G_V \langle n(z) \rangle = const.$
- phase diagram: [S. Carignano, D. Nickel, M.B., in prep.]

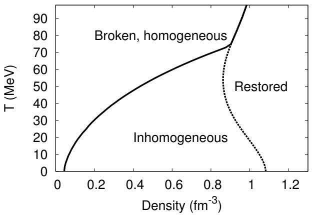


- inhomogeneous regime only streched in μ direction

Influence of vector interactions

- include vector interaction: $\mathcal{L}_V = -G_V(\bar{q}\gamma^\mu q)^2$
- approximation: $\tilde{\mu}(z) = \mu - 2G_V\, n(z) \;\rightarrow\; \mu - 2G_V \langle n(z)\rangle = const.$
- phase diagram: [S. Carignano, D. Nickel, M.B., in prep.]





  - inhomogeneous regime only streched in μ direction
  - T-$\langle n\rangle$ phase diagram independent of G_V!

PNJL MODEL

PNJL model

TECHNISCHE UNIVERSITÄT DARMSTADT

- main shortcoming of the NJL model: no confinement
  - unphysical $q\bar{q}$ decays of mesons
  - unphysical contribution of free quarks to the pressure at low T

PNJL model

- main shortcoming of the NJL model: no confinement
  - unphysical $q\bar{q}$ decays of mesons
  - unphysical contribution of free quarks to the pressure at low T
- Polyakov loop: $\ell = \frac{1}{N_c}\left\langle \mathrm{Tr}\,\mathcal{P}\exp\left[-i\int_0^\beta dx_4\, A_4(x_4,\vec{x})\right]\right\rangle$
  - order parameter for confinement (at infinite quark mass):
    - $\ell = 0$ confined
    - $\ell \neq 0$ deconfined

PNJL model

TECHNISCHE UNIVERSITÄT DARMSTADT

- main shortcoming of the NJL model: no confinement
  - unphysical $q\bar{q}$ decays of mesons
  - unphysical contribution of free quarks to the pressure at low T
- Polyakov loop: $\ell = \frac{1}{N_c}\langle \mathrm{Tr}\, \mathcal{P} \exp \left[-i \int_0^\beta dx_4\, A_4(x_4, \vec{x}) \right] \rangle$
  - order parameter for confinement (at infinite quark mass):
    - $\ell = 0$ confined
    - $\ell \neq 0$ deconfined
- P(olyakov loop enhanced) NJL model: [K. Fukushima, PLB (2004)]

$$\mathcal{L}_{PNJL} = \bar{q}(i\not{D} - m)q + G\left[(\bar{q}q)^2 + (\bar{q}i\gamma_5\vec{\tau}q)^2\right] - \mathcal{U}(\ell, \bar{\ell})$$

  - covariant derivative: $D_\mu = \partial_\mu - iA_\mu$, $A_\mu = \delta^0_\mu A_0$ constant background field
  - $\mathcal{U}(\ell, \bar{\ell})$ phenomenological potential

PNJL model: thermodynamics

TECHNISCHE UNIVERSITÄT DARMSTADT

- thermodynamic potential (thermal quark part):

$$\Omega_{q,th} = -2N_f T \int \frac{d^3p}{(2\pi)^3} \Big\{ \ln\Big(1 + 3\,\ell\, e^{-(E_p-\mu)/T} + 3\,\bar{\ell}\, e^{-2(E_p-\mu)/T} + e^{-3(E_p-\mu)/T}\Big) + \ln\Big(1 + 3\,\bar{\ell}\, e^{-(E_p+\mu)/T} + 3\,\ell\, e^{-2(E_p+\mu)/T} + e^{-3(E_p+\mu)/T}\Big)\Big\}$$

PNJL model: thermodynamics

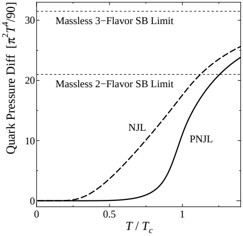
TECHNISCHE UNIVERSITÄT DARMSTADT

- thermodynamic potential (thermal quark part):

$$\Omega_{q,th} = -2N_f T \int \frac{d^3p}{(2\pi)^3} \Big\{ \ln\Big(1 + 3\,\ell\, e^{-(E_p-\mu)/T} + 3\,\bar{\ell}\, e^{-2(E_p-\mu)/T} + e^{-3(E_p-\mu)/T}\Big) + \ln\Big(1 + 3\,\bar{\ell}\, e^{-(E_p+\mu)/T} + 3\,\ell\, e^{-2(E_p+\mu)/T} + e^{-3(E_p+\mu)/T}\Big)\Big\}$$

- thermal quarks strongly suppressed for $\ell = \bar{\ell} = 0$

pressure



[K. Fukushima, PRD (2008)]

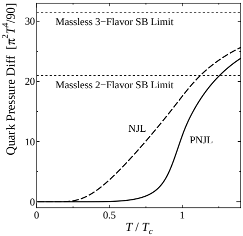
PNJL model: thermodynamics

- thermodynamic potential (thermal quark part):

$$\Omega_{q,th} = -2N_f T \int \frac{d^3p}{(2\pi)^3} \Big\{ \ln\Big(1 + 3\,\ell\, e^{-(E_p-\mu)/T} + 3\,\bar{\ell}\, e^{-2(E_p-\mu)/T} + e^{-3(E_p-\mu)/T}\Big) + \ln\Big(1 + 3\,\bar{\ell}\, e^{-(E_p+\mu)/T} + 3\,\ell\, e^{-2(E_p+\mu)/T} + e^{-3(E_p+\mu)/T}\Big)\Big\}$$

- thermal quarks strongly suppressed for $\ell = \bar{\ell} = 0$ (but $\bar{q}q$ decays of mesons still possible [Hansen et al., PRD '07])

pressure



[K. Fukushima, PRD (2008)]

PNJL model: thermodynamics

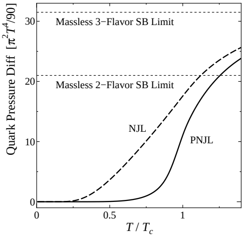
TECHNISCHE UNIVERSITÄT DARMSTADT

- thermodynamic potential (thermal quark part):

$$\Omega_{q,th} = -2N_f T \int \frac{d^3p}{(2\pi)^3} \Big\{ \ln\Big(1 + 3\,\ell\, e^{-(E_p-\mu)/T} + 3\,\bar\ell\, e^{-2(E_p-\mu)/T} + e^{-3(E_p-\mu)/T}\Big) + \ln\Big(1 + 3\,\bar\ell\, e^{-(E_p+\mu)/T} + 3\,\ell\, e^{-2(E_p+\mu)/T} + e^{-3(E_p+\mu)/T}\Big)\Big\}$$

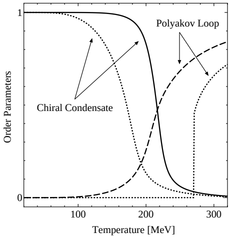
- thermal quarks strongly suppressed for $\ell = \bar\ell = 0$ (but $\bar q q$ decays of mesons still possible [Hansen et al., PRD '07])
- chiral and deconfinement transitions (partially) synchronized

pressure



[K. Fukushima, PRD (2008)]

$\langle \bar q q \rangle$, $\ell = \bar\ell$



[K. Fukushima, PLB (2004)]

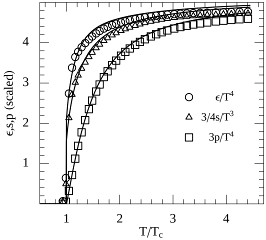
Comparison with lattice results

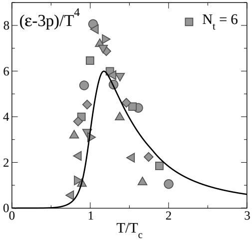
TECHNISCHE UNIVERSITÄT DARMSTADT

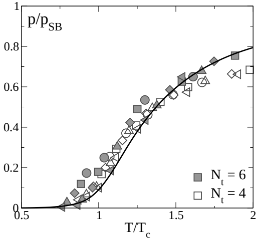
- strategy:
  - Polyakov-loop parameters fitted in the pure gauge sector
  - NJL parameters fitted in vacuum
  - predictions for the quark sector in medium

Comparison with lattice results

- strategy:
  - Polyakov-loop parameters fitted in the pure gauge sector
  - NJL parameters fitted in vacuum
  - predictions for the quark sector in medium
- results:





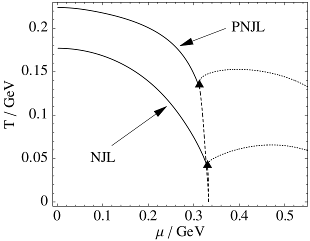


[C. Ratti, S. Roessner, M.A. Thaler, W. Weise, EPJ C (2007); [C. Ratti, . A. Thaler, W. Weise, PRD (2006); lattice: A. Ali Khan et al., PRD (2001)]

lattice: G. Boyd et al., NPB (1996)]

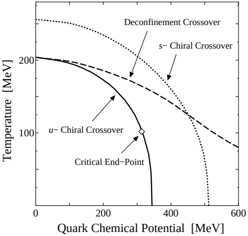
Phase diagrams

- two-flavor model with CSC:



[S. Rößner, C. Ratti, W. Weise, PRD (2007)]

- three-flavor model:



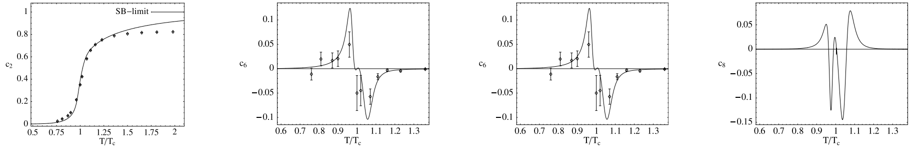
[K. Fukushima, PRD (2008)]

- suppression of quark d.o.f. ➜ critical temperatures shifted to higher values
- chiral and “deconfinement” transition do not necessarily stay together (➜ quarkyonic phase?)

Comparison with lattice results: techniques for non-vanishing chemical potential

TECHNISCHE UNIVERSITÄT DARMSTADT

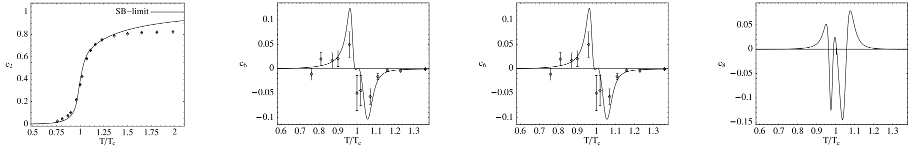
- Taylor expansion of the pressure: $\frac{p}{T^4}(T, \mu) = \sum_{n=0}^{\infty} c_n(T) \left(\frac{\mu}{T}\right)^n,$



[S. Rößner, C. Ratti, W. Weise, PRD (2007); lattice: C.R. Allton et al., PRD (2002,2003)]

Comparison with lattice results: techniques for non-vanishing chemical potential

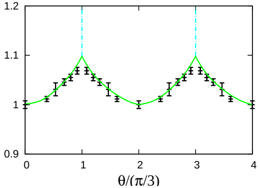
- Taylor expansion of the pressure: $\frac{p}{T^4}(T,\mu) = \sum_{n=0}^{\infty} c_n(T) \left(\frac{\mu}{T}\right)^n$,



[S. Rößner, C. Ratti, W. Weise, PRD (2007); lattice: C.R. Allton et al., PRD (2002,2003)]

- imaginary chemical potential:
  - $\mu = i\mu_I = i\theta\, T$
  - periodicity: $\theta \longrightarrow \theta + \frac{2\pi k}{3}$

  [A. Roberge, N. Weiss, NPB (1986)]



[Y. Sakai et al., PRD (2009) lattice: L.K. Wu, X.Q. Luo, H.S. Chen, PRD (2007)]

PNJL beyond mean-field approximation

TECHNISCHE UNIVERSITÄT DARMSTADT

- PNJL at mean field: quarks suppressed at low T, but no hadrons either ...

➜ include meson contributions!

PNJL beyond mean-field approximation

TECHNISCHE
UNIVERSITÄT
DARMSTADT

- PNJL at mean field: quarks suppressed at low T, but no hadrons either ...

➜ include meson contributions!

- ring sum: Ω_{ring} =  (NLO in a $1/N_c$ expansion)

PNJL beyond mean-field approximation

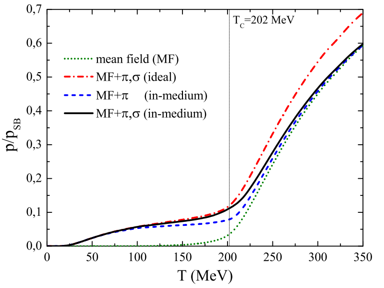
TECHNISCHE UNIVERSITÄT DARMSTADT

- PNJL at mean field: quarks suppressed at low T, but no hadrons either ...

➜ include meson contributions!

- ring sum: Ω_{ring} =  (NLO in a $1/N_c$ expansion)

- pressure: [Blaschke, M.B., Radzhabov, Volkov, Yad. Fiz. (2008)]



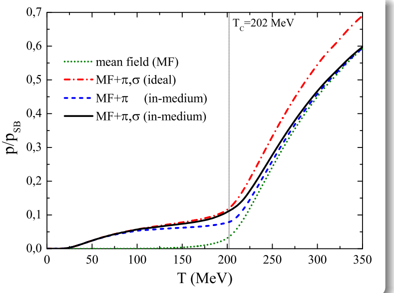
PNJL beyond mean-field approximation

TECHNISCHE UNIVERSITÄT DARMSTADT

- PNJL at mean field: quarks suppressed at low T, but no hadrons either ...

➜ include meson contributions!

- ring sum: Ω_{ring} =  (NLO in a $1/N_c$ expansion)
- pressure: [Blaschke, M.B., Radzhabov, Volkov, Yad. Fiz. (2008)]



- $T \lesssim T_c$: dominated by mesons

PNJL beyond mean-field approximation

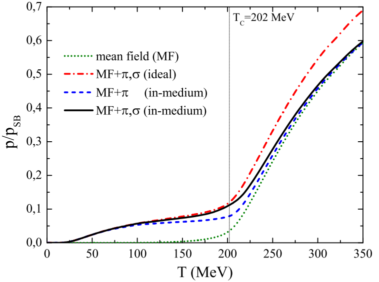
TECHNISCHE UNIVERSITÄT DARMSTADT

- PNJL at mean field: quarks suppressed at low T, but no hadrons either ...

➜ include meson contributions!

- ring sum: Ω_{ring} =  (NLO in a $1/N_c$ expansion)

- pressure: [Blaschke, M.B., Radzhabov, Volkov, Yad. Fiz. (2008)]



- $T \lesssim T_c$: dominated by mesons
- $T \lesssim 100$ MeV: almost ideal pion gas

PNJL beyond mean-field approximation

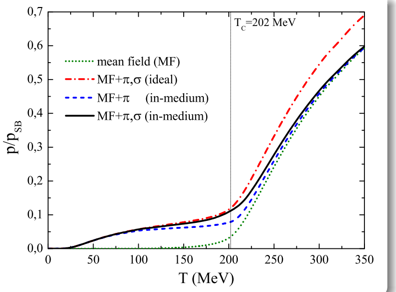
TECHNISCHE UNIVERSITÄT DARMSTADT

- PNJL at mean field: quarks suppressed at low T, but no hadrons either ...

➜ include meson contributions!

- ring sum: Ω_{ring} =  (NLO in a $1/N_c$ expansion)

- pressure: [Blaschke, M.B., Radzhabov, Volkov, Yad. Fiz. (2008)]



- $T \lesssim T_c$:
  dominated by mesons
- $T \lesssim 100$ MeV:
  almost ideal pion gas
- $T > T_c$:
  gradual convergence to mean field

Literature

- NJL reviews:

  - U. Vogl and W. Weise,
    "The Nambu and Jona Lasinio model: Its implications for hadrons and nuclei,"
    Prog. Part. Nucl. Phys. **27**, 195 (1991).
  - S. P. Klevansky,
    "The Nambu-Jona-Lasinio model of quantum chromodynamics,"
    Rev. Mod. Phys. **64**, 649 (1992).
  - T. Hatsuda and T. Kunihiro,
    "QCD phenomenology based on a chiral effective Lagrangian,"
    Phys. Rept. **247**, 221 (1994) [arXiv:hep-ph/9401310].
  - M. Buballa,
    "NJL model analysis of dense quark matter,"
    Phys. Rept. **407**, 205 (2005) [arXiv:hep-ph/0402234].

Literature

TECHNISCHE UNIVERSITÄT DARMSTADT

- NJL reviews:
  - U. Vogl and W. Weise,
    "The Nambu and Jona Lasinio model: Its implications for hadrons and nuclei,"
    Prog. Part. Nucl. Phys. **27**, 195 (1991).
  - S. P. Klevansky,
    "The Nambu-Jona-Lasinio model of quantum chromodynamics,"
    Rev. Mod. Phys. **64**, 649 (1992).
  - T. Hatsuda and T. Kunihiro,
    "QCD phenomenology based on a chiral effective Lagrangian,"
    Phys. Rept. **247**, 221 (1994) [arXiv:hep-ph/9401310].
  - M. Buballa,
    "NJL model analysis of dense quark matter,"
    Phys. Rept. **407**, 205 (2005) [arXiv:hep-ph/0402234].
- color superconductivity:
  - M. G. Alford, A. Schmitt, K. Rajagopal and T. Schäfer,
    "Color superconductivity in dense quark matter,"
    Rev. Mod. Phys. **80**, 1455 (2008) [arXiv:0709.4635 [hep-ph]].

Literature

TECHNISCHE UNIVERSITÄT DARMSTADT

- NJL reviews:
  - U. Vogl and W. Weise,
    "The Nambu and Jona Lasinio model: Its implications for hadrons and nuclei,"
    Prog. Part. Nucl. Phys. **27**, 195 (1991).
  - S. P. Klevansky,
    "The Nambu-Jona-Lasinio model of quantum chromodynamics,"
    Rev. Mod. Phys. **64**, 649 (1992).
  - T. Hatsuda and T. Kunihiro,
    "QCD phenomenology based on a chiral effective Lagrangian,"
    Phys. Rept. **247**, 221 (1994) [arXiv:hep-ph/9401310].
  - M. Buballa,
    "NJL model analysis of dense quark matter,"
    Phys. Rept. **407**, 205 (2005) [arXiv:hep-ph/0402234].
- color superconductivity:
  - M. G. Alford, A. Schmitt, K. Rajagopal and T. Schäfer,
    "Color superconductivity in dense quark matter,"
    Rev. Mod. Phys. **80**, 1455 (2008) [arXiv:0709.4635 [hep-ph]].
- no review on PNJL so far

THANK YOU!