Heavy Quarks at High Temperature

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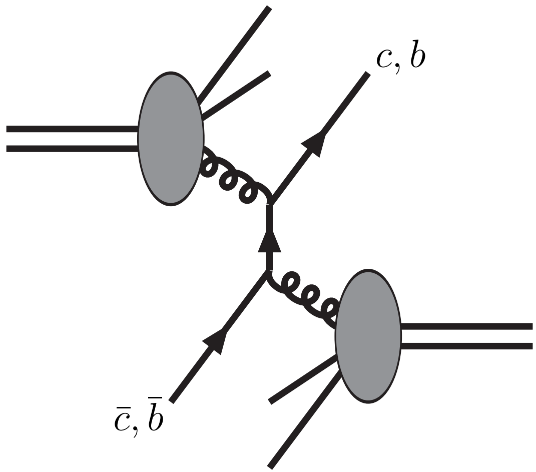
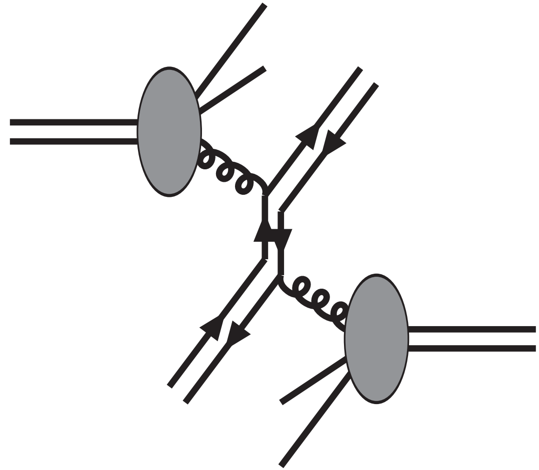
Initial production



Heavy Quarks

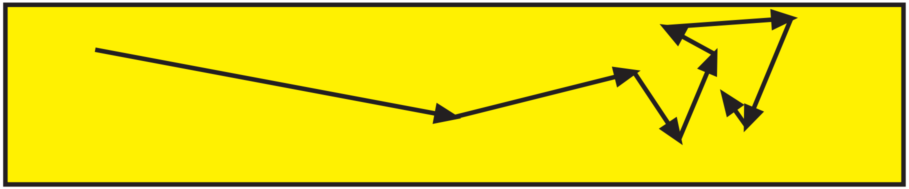


Heavy Quarkonium

Subsequently heavy probes **propagate** through a soft "medium".

In the end they **decay**; heavy quarks often as $c \longrightarrow \ell \nu X$; heavy quarkonium often as $c\bar{c} \longrightarrow \ell^+ \ell^-$; the leptons ℓ can be observed.

Propagation through the medium

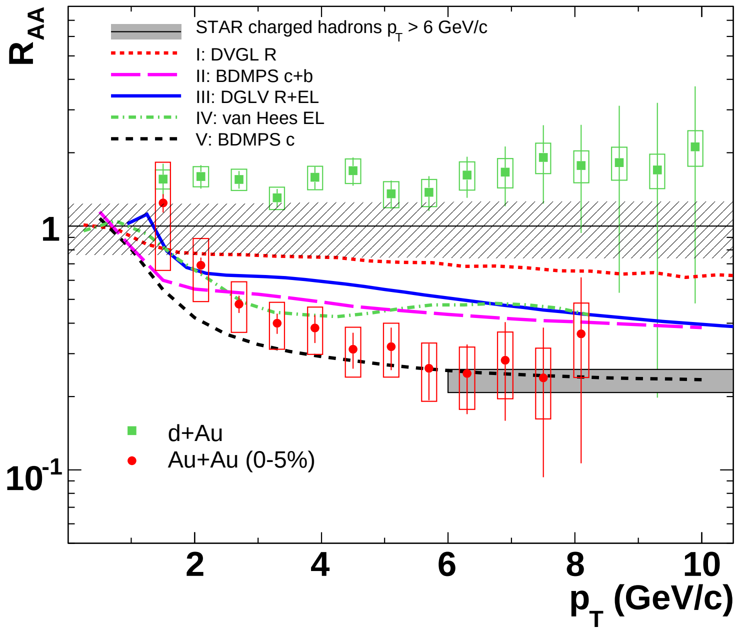


Like heavy particles in Brownian motion, heavy quark jets ("open charm") tend to get stopped ("quenched") by scatterings.

Observables characterizing the stopping are referred to as the momentum diffusion coefficient / the "drag force" / the kinetic thermalization rate.

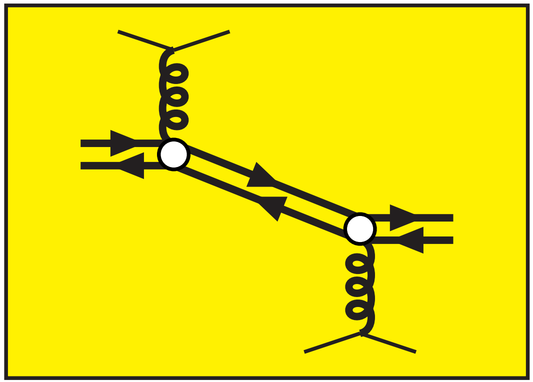
Indeed in Au + Au less ℓ observed than expected:

STAR nucl-ex/0607012, PHENIX nucl-ex/0611018



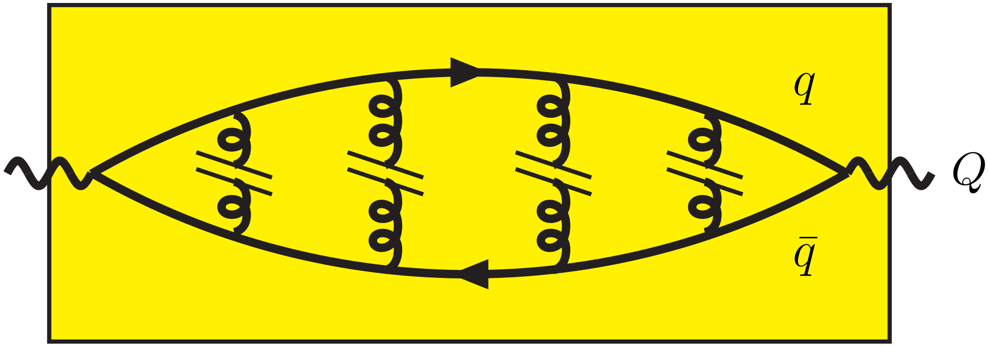
Heavy quarkonium feels less drag than heavy quarks because it has no net colour charge.

However, because of its finite size, it does have a **colour dipole** which leads to kicks and an eventual "decoherence" of its state:



But these effects should be suppressed by $\mathcal{O}(rT)^2$?

On the other hand the Coulomb potential gets Debye-screened within the medium:



So the effective r could be larger than at $T = 0$, and the thermal effects on quarkonium propagation significant.

These effects make themselves visible in the shape of the quarkonium **spectral function**, which determines the thermal component in the dilepton production rate.

To keep in mind:

Propagation takes place in Minkowskian time (t), with a Minkowskian frequency (ω), but within a thermal system ($\beta = 1/T$).

Formally:

$$\left\langle \frac{1}{2} [\hat{\mathcal{J}}^{\mu}(t, \mathbf{x}), \hat{\mathcal{J}}_{\mu}(0, \mathbf{0})] \right\rangle \ ,$$

$$\hat{\mathcal{J}}^{\mu}(t, \mathbf{x}) = e^{i\hat{H}t} \hat{\mathcal{J}}^{\mu}(0, \mathbf{x}) e^{-i\hat{H}t} \ ,$$

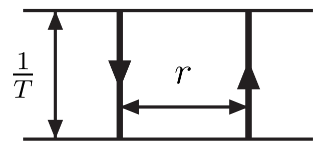
$$\langle \ldots \rangle \equiv \frac{1}{\mathcal{Z}} \mathrm{Tr} \left[(\ldots) e^{-\beta \hat{H}} \right] \ .$$

These are technically challenging observables!

I. To which extent is the system perturbative?

A popular lattice observable (having perhaps something to do with heavy quarkonium):

$$\psi_{\rm C}(r) \equiv \frac{1}{N_{\rm c}} \langle {\rm Tr}[P_{\bf r} P_{\bf 0}^\dagger] \rangle_{\text{Coulomb}} \; .$$



No real time here!

(Gauge invariant alternatives, $\psi_{\rm W}(r) \equiv \frac{1}{N_{\rm c}} \langle {\rm Tr}[P_{\bf r} W_0 P_{\bf 0}^\dagger W_0^\dagger] \rangle$ or $\psi_{\rm T}(r) \equiv \frac{1}{N_{\rm c}^2} \langle {\rm Tr}[P_{\bf r}]\, {\rm Tr}[P_{\bf 0}^\dagger] \rangle$, do **not** reduce to the known zero-temperature static potential at short distances.)

Perturbative expression recently worked out up to $\mathcal{O}(\alpha_s^2)$:

Burnier et al 0911.3480

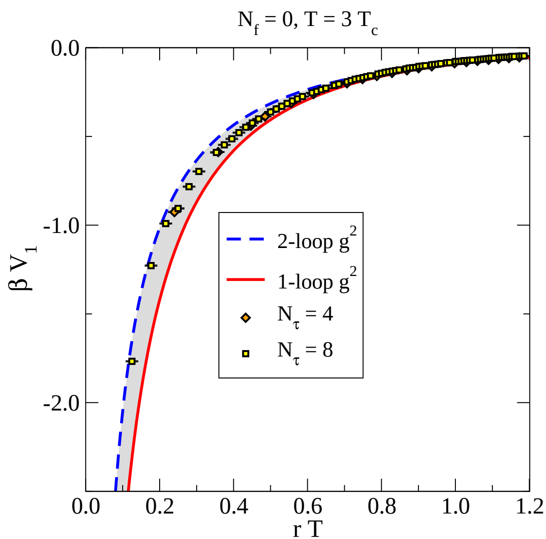
$$\ln\left(\frac{\psi_{\rm C}(r)}{|\psi_{\rm P}|^2}\right) \approx \frac{g^2 C_F \exp(-m_{\rm E} r)}{4\pi T r}\left\{1 + \frac{g^2}{(4\pi)^2}\left[\frac{11N_{\rm c}}{3}\left(L_{\rm b}+1\right) - \frac{2N_{\rm f}}{3}\left(L_{\rm f}-1\right)\right]\right\}$$
$$+ \frac{g^4 C_F N_{\rm c}\exp(-m_{\rm E} r)}{(4\pi)^2}\left[2 - \ln(2m_{\rm E} r) - \gamma_{\rm E} + e^{2m_{\rm E} r}E_1(2m_{\rm E} r)\right] - \frac{g^4 C_F N_{\rm c}}{(4\pi)^2}\frac{\exp(-2m_{\rm E} r)}{8T^2 r^2}$$
$$+ \frac{g^4 C_F N_{\rm c}}{(4\pi)^2}\left[\frac{1}{12T^2r^2} + \frac{{\rm Li}_2(e^{-4\pi T r})}{(2\pi T r)^2} + \frac{1}{\pi T r}\int_1^\infty {\rm d}x\left(\frac{1}{x^2} - \frac{1}{2x^4}\right)\ln\left(1 - e^{-4\pi T r x}\right)\right]$$
$$+ \frac{g^4 C_F N_{\rm f}}{(4\pi)^2}\left[\frac{1}{2\pi T r}\int_1^\infty {\rm d}x\left(\frac{1}{x^2} - \frac{1}{x^4}\right)\ln\frac{1+e^{-2\pi T r x}}{1-e^{-2\pi T r x}}\right] + \mathcal{O}(g^5)\ ,$$

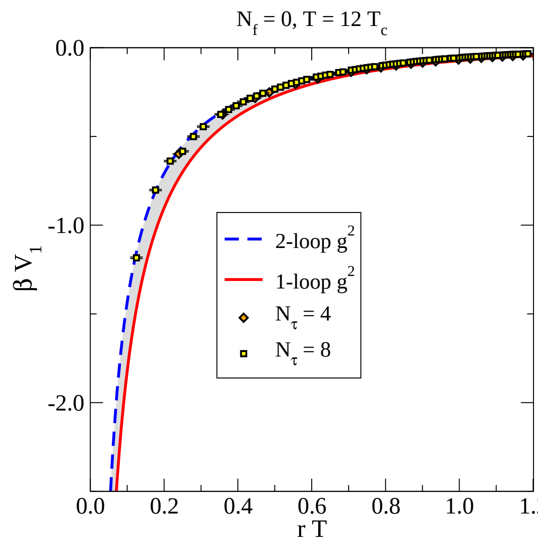
For $r \ll \frac{1}{\pi T}$, this reproduces the classic $T = 0$ static potential.

Fischler NPB 129 (1977) 157

Comparison with lattice $[\beta V_1 \equiv -\ln(\psi_{\rm C}/|\psi_{\rm P}|^2)]$:

data from Kaczmarek et al hep-lat/0207002

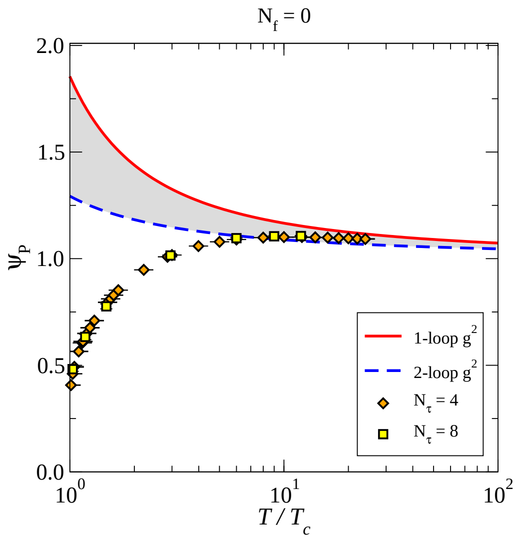




So, somewhat surprisingly, we find reasonable qualitative agreement even at low temperatures ($T_{\rm c} \approx 200$ MeV).

(Match is often worse for observables with no perturbative $T = 0$ part, like the expectation value of a single Polyakov loop.)

$$\psi_{\rm P} = \frac{1}{N_{\rm c}}\langle {\rm Tr}\,[P_{\mathbf{r}}]\rangle = 1 + \frac{g^2 C_F m_{\rm E}}{8\pi T} + \frac{g^4 C_F}{(4\pi)^2}\left[N_{\rm f}\left(-\frac{\ln 2}{2}\right) + N_{\rm c}\left(\ln\frac{m_{\rm E}}{T} + \frac{1}{4}\right)\right] + \mathcal{O}(g^5)\ .$$



data from Gupta et al 0711.2251

II. How to really address heavy quarkonium?

In order to get a handle on the many scales appearing, both vacuum and thermal, need to make use of **effective field theories**.

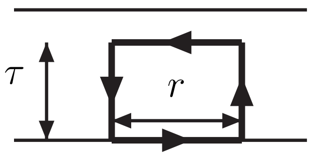
For quarkonium, the relevant framework is that of **NRQCD** (Non-Relativistic QCD) or one of its descendants (pNRQCD etc).

In such frameworks, various heavy quark potentials appear as **matching coefficients**, and can be given a concrete definition (at least within perturbation theory, which now appears reasonable).

in the thermal context: ML et al hep-ph/0611300; Beraudo et al 0712.4394; Escobedo Soto 0804.0691; Brambilla et al 0804.0993

It is important to keep in mind that the Euclidean $\beta = 1/T$ is "small", while the Minkowskian t is "large" in the "static" limit.

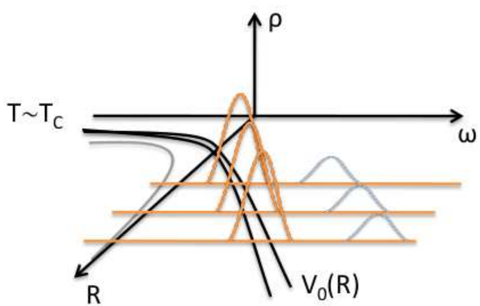
$\Rightarrow$ define a potential from analytic continuation:



$$C_E(\tau, r) \equiv \langle \mathrm{Tr}[W_E(\tau, r)] \rangle \ ,$$
$$i\partial_t C_E(it, r) \equiv V_>(t, r) C_E(it, r) \ .$$

The static limit $V_>(\infty, r)$ through **spectral analysis**.

Rothkopf Hatsuda Sasaki 0910.2321



Position of the spectral peak: average energy, $\mathrm{Re}\, V_>(\infty, r)$.
Its width: decoherence, $\mathrm{Im}\, V_>(\infty, r)$.

Explicitly at $\mathcal{O}(\alpha_s)$:

$$\mathrm{Re}\, V_>(\infty, r) = -\frac{g^2 C_F}{4\pi}\left[m_{\mathsf{E}} + \frac{\exp(-m_{\mathsf{E}} r)}{r}\right] ,$$

$$\mathrm{Im}\, V_>(\infty, r) = -\frac{g^2 T C_F}{4\pi}\,\phi(m_{\mathsf{E}} r) ,$$

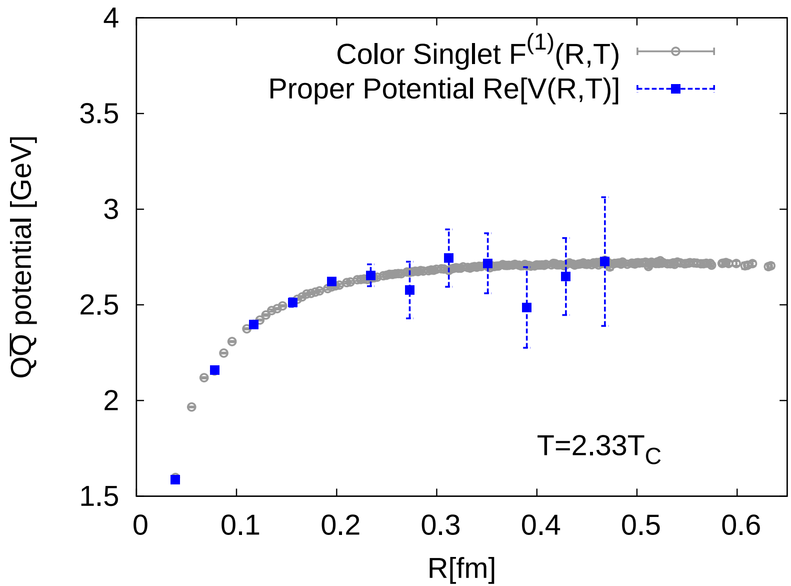
where $m_{\mathsf{E}} \sim gT$ is the Debye mass, $C_F \equiv 4/3$, and

$$\phi(x) = 2\int_0^\infty \frac{\mathrm{d}z\, z}{(z^2+1)^2}\left[1 - \frac{\sin(zx)}{zx}\right] .$$

At $m_{\mathsf{E}} r \ll 1$, $\phi \sim (m_{\mathsf{E}} r)^2$, as is appropriate for a dipole.

Numerical test for the **real part**:

Rothkopf Hatsuda Sasaki 0910.2321



Low and behold, it does appear to agree with the Coulomb gauge potential! But more precision required for definite conclusions.

III. How to address single heavy quarks?

In this case the relevant effective framework is that of **HQET** (Heavy Quark Effective Theory).

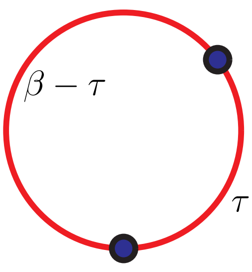
Taking further steps within that theory, the heavy quark kinetic thermalization rate can further be reduced to a **purely gluonic observable** — in analogy with the reduction of heavy quarkonium properties to static potentials within NRQCD!

More precisely, the thermalization rate, η, can be fluctuation-dissipation-related to a force-force transport coefficient, κ:

$$\eta = \frac{\kappa}{2M_{\text{kin}}T}\left(1 + O\left(\frac{\alpha_s^{3/2}T}{M_{\text{kin}}}\right)\right) , \quad \kappa = \lim_{\omega\to 0}\frac{2T\rho_E(\omega)}{\omega} .$$

Here ρ_E is the spectral function corresponding to the Euclidean correlator

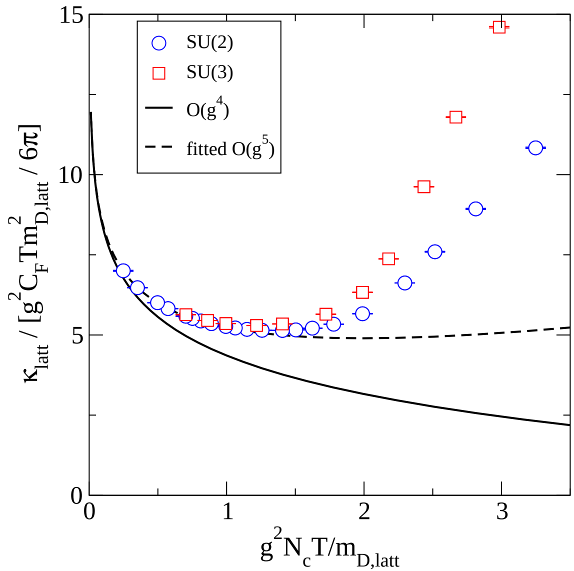
$$G_E(\tau) = -\frac{1}{3}\sum_{i=1}^{3}\frac{\langle \mathrm{Re\,Tr}[U_{\beta;\tau}\, gE_i(\tau,\mathbf{0})\, U_{\tau;0}\, gE_i(0,\mathbf{0})]\rangle}{\langle \mathrm{Re\,Tr}[U_{\beta;0}]\rangle} .$$



Casalderrey-Solana Teaney hep-ph/0605199;

Caron-Huot ML Moore 0901.1195

So far κ has only been measured within "classical lattice gauge theory", but the result is interesting if plotted in terms of a quantity having an analogue in QCD, $m^2_{\text{D,latt}} \sim g^2 T/a$:



$O(g^5)$ in QCD: Caron-Huot Moore, 0708.4232

data: ML Moore Philipsen Tassler, 0902.2856

$\Rightarrow$ the kinetic thermalization rate could be unexpectedly large — as appears to be required by phenomenology.

Phenomenological values:

$$\eta_{\rm pQCD} \sim \frac{g^4}{8\pi}\frac{T^2}{M_{\sf kin}} \sim 0.3\frac{T^2}{M_{\sf kin}}$$

Braaten Thoma PRD 44 (1991) 1298, 2625; Moore Teaney hep-ph/0412346

$$\eta_{\rm exp} \sim (1...3) \times \frac{T^2}{M_{\sf kin}} \quad \text{(no } \chi^2\text{-minimization though)}$$

e.g. Akamatsu et al 0809.1499

$$\eta_{\rm AdS/CFT} \sim \frac{g\pi\sqrt{3}}{2}\frac{T^2}{M_{\sf kin}} \sim 4\frac{T^2}{M_{\sf kin}}$$

Herzog et al hep-th/0605158; Gubser hep-th/0605182; Casalderrey Teaney hep-ph/0605199

So, it would be interesting to determine κ & η from lattice-QCD!

IV. Basics of spectral functions @ Euclidean lattice

$\hat{O} \equiv$ spatial component of a conserved current ($\hat{T}^{\mu i}$, $\hat{J}^{i}$).

Heisenberg picture:

$$\hat{O}(t) = e^{i\hat{H}t}\hat{O}(0)e^{-i\hat{H}t} \ .$$

Spectral function:

$$\rho(\omega) = \int_{-\infty}^{\infty} \mathrm{d}t \, e^{i\omega t} \frac{1}{\mathcal{Z}} \, \mathrm{Tr} \left\{ e^{-\beta\hat{H}} \frac{1}{2} \left[\hat{O}(t), \hat{O}(0) \right] \right\} \ .$$

Transport coefficient: $\lim_{\omega \to 0} \frac{\rho(\omega)}{\omega}$.

This yields heat conductivity ($\hat{T}^{0i}$), electrical conductivity ($\hat{J}^{i}_{\text{em}}$), shear viscosity ($\hat{T}^{ji}$), bulk viscosity ($\hat{T}^{ii}$), flavour diffusion coefficient ($\hat{J}^{i}_{\text{f}}$), particle production rate ($\hat{J}^{i}_{\text{em}}$), . . .

Some gymnastics with Green's functions

Apart from the "real-time" Heisenberg-operators

$$\hat{A}(t) = e^{i\hat{H}t}\hat{A}(0)e^{-i\hat{H}t} \ ,$$

we define "imaginary-time" Heisenberg-operators as

$$\hat{A}(\tau) \equiv e^{\hat{H}\tau}\hat{A}(0)e^{-\hat{H}\tau} \ .$$

Note, however, that $\tau \in \mathbb{R}$ in the latter case as well; in fact, we will always restrict to

$$0 \leq \tau \leq \beta \ .$$

Then we define:

$$\Pi_>(\omega) \equiv \int_{-\infty}^{\infty} \mathrm{d}t\, e^{i\omega t} \langle \hat{A}(t)\hat{B}(0)\rangle \ ,$$

$$\Pi_<(\omega) \equiv \int_{-\infty}^{\infty} \mathrm{d}t\, e^{i\omega t} \langle \hat{B}(0)\hat{A}(t)\rangle \ ,$$

$$\rho(\omega) \equiv \int_{-\infty}^{\infty} \mathrm{d}t\, e^{i\omega t} \left\langle \frac{1}{2}[\hat{A}(t), \hat{B}(0)] \right\rangle \ ,$$

$$G_E(\tau) \equiv \langle \hat{A}(\tau)\hat{B}(0)\rangle \ ,$$

$$\tilde{G}_E(\omega_n^{\mathsf{b}}) \equiv \int_0^{\beta} \mathrm{d}\tau\, e^{i\omega_n^{\mathsf{b}}\tau} G_E(\tau) \ ; \quad \omega_n^{\mathsf{b}} \equiv 2\pi n T \ , \quad n \in \mathbb{Z} \ .$$

Here the expectation value is $\langle \ldots \rangle \equiv \frac{1}{\mathcal{Z}} \, \mathrm{Tr} \left\{ e^{-\beta \hat{H}} (\ldots) \right\}$.

Apart from relations between the Minkowskian objects that can be established by inserting $\mathbb{1} = \sum_n |n\rangle\langle n|$, *viz.*

$$\Pi_<(\omega) \;=\; \frac{2}{e^{\beta\omega}-1}\rho(\omega) = 2n_\mathrm{B}(\omega)\rho(\omega)\ ,$$

$$\Pi_>(\omega) \;=\; \frac{2}{1-e^{-\beta\omega}}\rho(\omega) = 2e^{\beta\omega}n_\mathrm{B}(\omega)\rho(\omega)\ ,$$

we need the following relations to the Euclidean ones:

$$\tilde{G}_E(\omega_n^\mathrm{b}) \;=\; \int_{-\infty}^{\infty}\frac{\mathrm{d}\omega}{\pi}\frac{\rho(\omega)}{\omega - i\omega_n^\mathrm{b}} \quad \text{“spectral representation”}\ ,$$

$$\rho(\omega) \;=\; \frac{1}{2i}\left\{\tilde{G}_E\left(-i[\omega+i0^+]\right) - \tilde{G}_E\left(-i[\omega-i0^+]\right)\right\}\ ,$$

$$G_E(\tau) \;=\; \int_0^{\infty}\frac{\mathrm{d}\omega}{\pi}\rho(\omega)\frac{\cosh\left(\frac{\beta}{2}-\tau\right)\omega}{\sinh\frac{\beta\omega}{2}}\ .$$

Proof of the “spectral representation”:

$$\begin{aligned}
\tilde{G}_E(\omega_n^{\mathsf{b}}) &= \int_0^\beta \mathrm{d}\tau\, e^{i\omega_n^{\mathsf{b}}\tau} \Big[\int_{-\infty}^{\infty} \frac{\mathrm{d}\omega}{2\pi}\, e^{-i\omega t}\Pi_>(\omega)\Big]_{it\to\tau} \\
&= \int_0^\beta \mathrm{d}\tau\, e^{i\omega_n^{\mathsf{b}}\tau} \int_{-\infty}^{\infty} \frac{\mathrm{d}\omega}{2\pi} e^{-\omega\tau}\Pi_>(\omega) \\
&= \int_0^\beta \mathrm{d}\tau\, e^{i\omega_n^{\mathsf{b}}\tau} \int_{-\infty}^{\infty} \frac{\mathrm{d}\omega}{2\pi} e^{-\omega\tau} \frac{2e^{\beta\omega}}{e^{\beta\omega}-1}\rho(\omega) \\
&= \int_{-\infty}^{\infty} \frac{\mathrm{d}\omega}{\pi} \frac{\rho(\omega)}{1-e^{-\beta\omega}} \left[\frac{e^{(i\omega_n^{\mathsf{b}}-\omega)\tau}}{i\omega_n^{\mathsf{b}}-\omega}\right]_0^\beta \\
&= \int_{-\infty}^{\infty} \frac{\mathrm{d}\omega}{\pi} \frac{\rho(\omega)}{1-e^{-\beta\omega}} \frac{e^{-\beta\omega}-1}{i\omega_n^{\mathsf{b}}-\omega} \\
&= \int_{-\infty}^{\infty} \frac{\mathrm{d}\omega}{\pi} \frac{\rho(\omega)}{\omega - i\omega_n^{\mathsf{b}}} \;.
\end{aligned}$$

To prove the relation

$$\rho(\omega) \;=\; \frac{1}{2i}\left\{\tilde{G}_E\left(-i[\omega + i0^+]\right) - \tilde{G}_E\left(-i[\omega - i0^+]\right)\right\} \;,$$

start from the spectral representation

$$\tilde{G}_E(\omega_n^{\mathsf{b}}) \;=\; \int_{-\infty}^{\infty}\frac{\mathrm{d}\omega}{\pi}\frac{\rho(\omega)}{\omega - i\omega_n^{\mathsf{b}}} \;,$$

and make use of

$$\frac{1}{x \pm i0^+} = P\left(\frac{1}{x}\right) \mp i\pi\delta(x) \;.$$

To prove the third relation, use the spectral representation:

$$
\begin{aligned}
G_E(\tau) &= T \sum_{\omega_n^{\mathsf{b}}} e^{-i\omega_n^{\mathsf{b}}\tau}\, \tilde{G}_E(\omega_n^{\mathsf{b}}) \\
&= T \sum_{\omega_n^{\mathsf{b}}} e^{-i\omega_n^{\mathsf{b}}\tau} \int_0^\infty \frac{\mathrm{d}\omega}{\pi} \left[\frac{\rho(\omega)}{\omega - i\omega_n^{\mathsf{b}}} + \frac{\rho(-\omega)}{-\omega - i\omega_n^{\mathsf{b}}} \right] \\
&= T \sum_{\omega_n^{\mathsf{b}}} e^{-i\omega_n^{\mathsf{b}}\tau} \int_0^\infty \frac{\mathrm{d}\omega}{\pi} \rho(\omega) \left[\frac{1}{\omega - i\omega_n^{\mathsf{b}}} + \frac{1}{\omega + i\omega_n^{\mathsf{b}}} \right] \\
&= \int_0^\infty \frac{\mathrm{d}\omega}{\pi} \rho(\omega)\; T \sum_{\omega_n^{\mathsf{b}}} e^{-i\omega_n^{\mathsf{b}}\tau} \frac{2\omega}{\omega^2 + (\omega_n^{\mathsf{b}})^2} \\
&= \int_0^\infty \frac{\mathrm{d}\omega}{\pi} \rho(\omega) \frac{\cosh\left(\frac{\beta}{2} - \tau\right)\omega}{\sinh \frac{\beta\omega}{2}} \, .
\end{aligned}
$$

We used $\rho(-\omega) = -\rho(\omega)$, true under weak assumptions.

To summarize, the spectral function $\rho(\omega)$ determines the Euclidean correlators, both in τ-space ($G_E(\tau)$) or in ω_n^{b}-space ($\tilde{G}_E(\omega_n^{\text{b}})$). The relations are **in principle** invertible.

This is conceptually comforting, because Euclidean observables can be computed with regular functional integrals, a well-defined procedure even on the non-perturbative (lattice) level.

The claim now is that for spatial components,

$$\frac{\rho_{ii}(\omega)}{\omega} \overset{\omega \ll T}{\approx} \Delta_{ii}(0) \frac{\beta\eta}{\omega^2 + \eta^2} \overset{\eta \to 0}{\approx} \Delta_{ii}(0)\beta\pi\delta(\omega) \ .$$

On the other hand for the zero components,

$$\frac{\rho_{00}(\omega)}{\omega} \overset{!}{=} \Delta_{00}(0)\beta\pi\delta(\omega) \ .$$

This means that

$$\begin{aligned} G^E_{\mu\nu}(\tau) &= \int_{-\infty}^{\infty} \frac{\mathrm{d}\omega}{2\pi} \rho_{\mu\nu}(\omega) \frac{\cosh\left(\frac{\beta}{2} - \tau\right)\omega}{\sinh\frac{\beta\omega}{2}} \\ &= \Delta_{\mu\nu}(0) \ , \end{aligned}$$

i.e. the correlator should be independent of τ!

Indeed, if we consider the correlator of a conserved charge in Euclidean spacetime, it must be a constant:

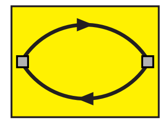
$$\partial_\tau \left\langle \int \mathrm{d}^3\mathbf{x}\, \hat{J}_0(\tau,\mathbf{x})\, \hat{J}_0(0,\mathbf{0}) \right\rangle = 0 \ .$$

In fact, for $\hat{J}_0 = \hat{\bar{\psi}}\gamma_0\hat{\psi}$, $T \ll M$, $g = 0$ (free limit),

$$\begin{aligned} G^E_{00}(\tau) &\equiv \left\langle \int \mathrm{d}^3\mathbf{x}\, \hat{J}_0(\tau,\mathbf{x})\, \hat{J}_0(0,\mathbf{0}) \right\rangle \\ &\approx 4N_\mathrm{c} \left(\frac{MT}{2\pi} \right)^{3/2} e^{-\beta M} \ . \end{aligned}$$

Consider then the correlator of the spatial components:

$$G^E_{ii}(\tau) \quad \equiv \quad \left\langle \int \mathrm{d}^3\mathbf{x}\, \hat{J}_i(\tau,\mathbf{x}) \hat{J}_i(0,\mathbf{0}) \right\rangle \; .$$

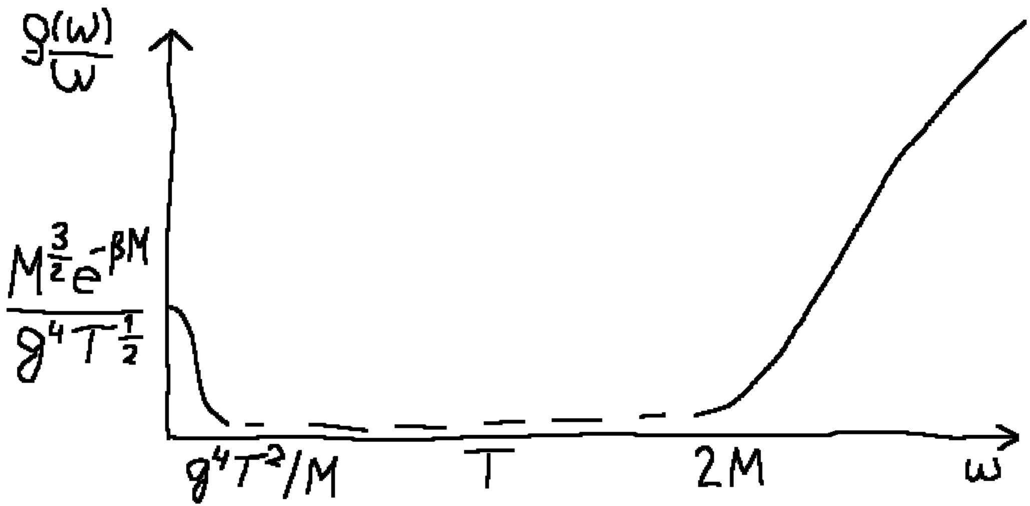
It turns out that in the free theory,  , this again contains a τ-independent "zero-mode":

$$\sum_{i=1}^{3} G^E_{ii}(\tau) = 4 N_\mathrm{c} \frac{3T}{M} \left(\frac{MT}{2\pi}\right)^{3/2} e^{-\beta M} + (\tau - \text{dep.}) \; .$$

However, now interactions can smoothen $\delta(\omega)$ from $\rho_{ii}(\omega)/\omega$. This yields a "**transport peak**".

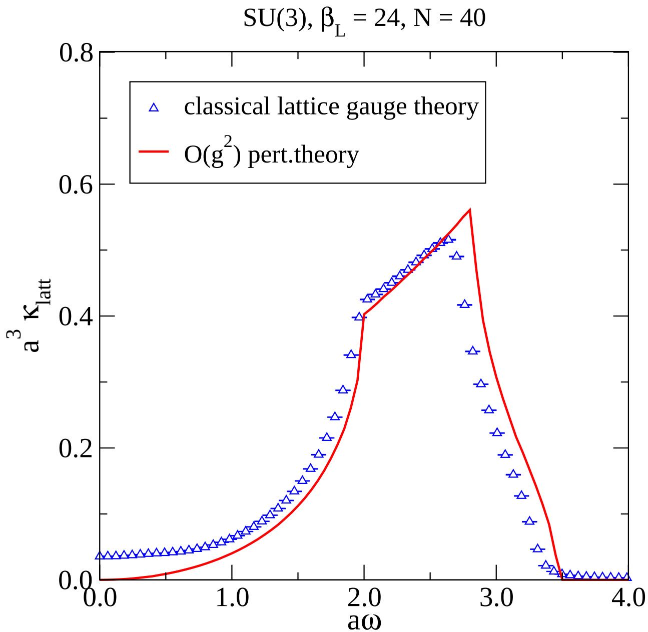
Summary

Generically, spectral functions corresponding to spatial components of conserved currents have **transport peaks** at small frequencies. In particular, for heavy quarks:



This makes a direct lattice determination **very** difficult!

But the reduction to the electric field correlator removes the problem! According to classical lattice gauge theory again:



The peak is absent because there is no conserved current left in the purely gluonic formulation, after integrating out the scale M.

Conclusions

Effective theory techniques allow to reduce heavy quark related "transport coefficients" to purely gluonic observables (static potential for quarkonium, electric correlator for single quarks).

The convergence of the weak-coupling expansion depends on the observable; it could be reasonable for static potentials.

Even for observables where perturbation theory is not applicable, like κ & η, the reduced formulation may lead to a more manageable lattice problem, if it can be justified non-perturbatively (as an expansion in T/M_{kin} rather than α_s).