The fluctuations of the quark number

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What: The distribution $\langle\delta(n-n')\rangle$ of n

Why: Understand how $\langle n\rangle$ builds up

How: Analytically in Chiral Perturbation Theory

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Why: Understand how $\langle n \rangle$ builds up

How: Analytically in Chiral Perturbation Theory

Shows: How Complex Langevin solves the sign problem

Pions have zero baryon charge

- so how can CPT teach us about n *?*

Certainly in CPT

$$\langle n \rangle = 0$$

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But $\langle \delta(n - n') \rangle$ is non trivial in CPT

Warning: $\langle\delta(n - n')\rangle$ is the distribution of n over A_ν

$$n \equiv \frac{d}{d\mu} \log\det(D + \mu\gamma_0 + m)$$

The average quark number, $\langle n_q \rangle$, is

$$\langle n_q \rangle \equiv \langle n \rangle$$

The average of the square of the quark number

$$\langle n_q^2 \rangle = \frac{1}{Z}\,\frac{d^2}{d\mu^2}\,Z = \langle n^2 \rangle + \langle \left(\frac{dn}{d\mu} \right) \rangle$$

The average of the square of n

$$\langle n^2 \rangle = \frac{1}{Z}\,\frac{d}{d\mu_u}\frac{d}{d\mu_d}\,Z\Big|_{\mu_u=\mu_d=\mu}$$

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$\langle n_q^2 \rangle$ not the second moment of $\langle \delta(n-n') \rangle$

Non trivial distribution from CPT

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1-loop free energy

$$G_0(\mu_1, \mu_2) = V \frac{m_\pi^2 T^2}{\pi^2} \sum_{n=1}^{\infty} \frac{K_2(\frac{m_\pi n}{T})}{n^2} \cosh(\frac{\mu_1 - \mu_2}{T} n)$$

$\langle \delta(n - n') \rangle$ shows the pion noise

$$\langle n \rangle = \int dn' \; n' \; \langle \delta(n - n') \rangle$$

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The sign problem and Complex Langevin

The distribution of n

$$n(\mu)^* = \left(\mathrm{Tr} \frac{\gamma_0}{D + \mu\gamma_0 + m} \right)^* = -n(-\mu)$$

Fluctuations in the complex n plane

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Fluctuations in the complex n plane

$$P_n(x,y) \equiv \left\langle \delta\left(x - \mathrm{Re}[n]\right)\delta\left(y - \mathrm{Im}[n]\right)\right\rangle$$

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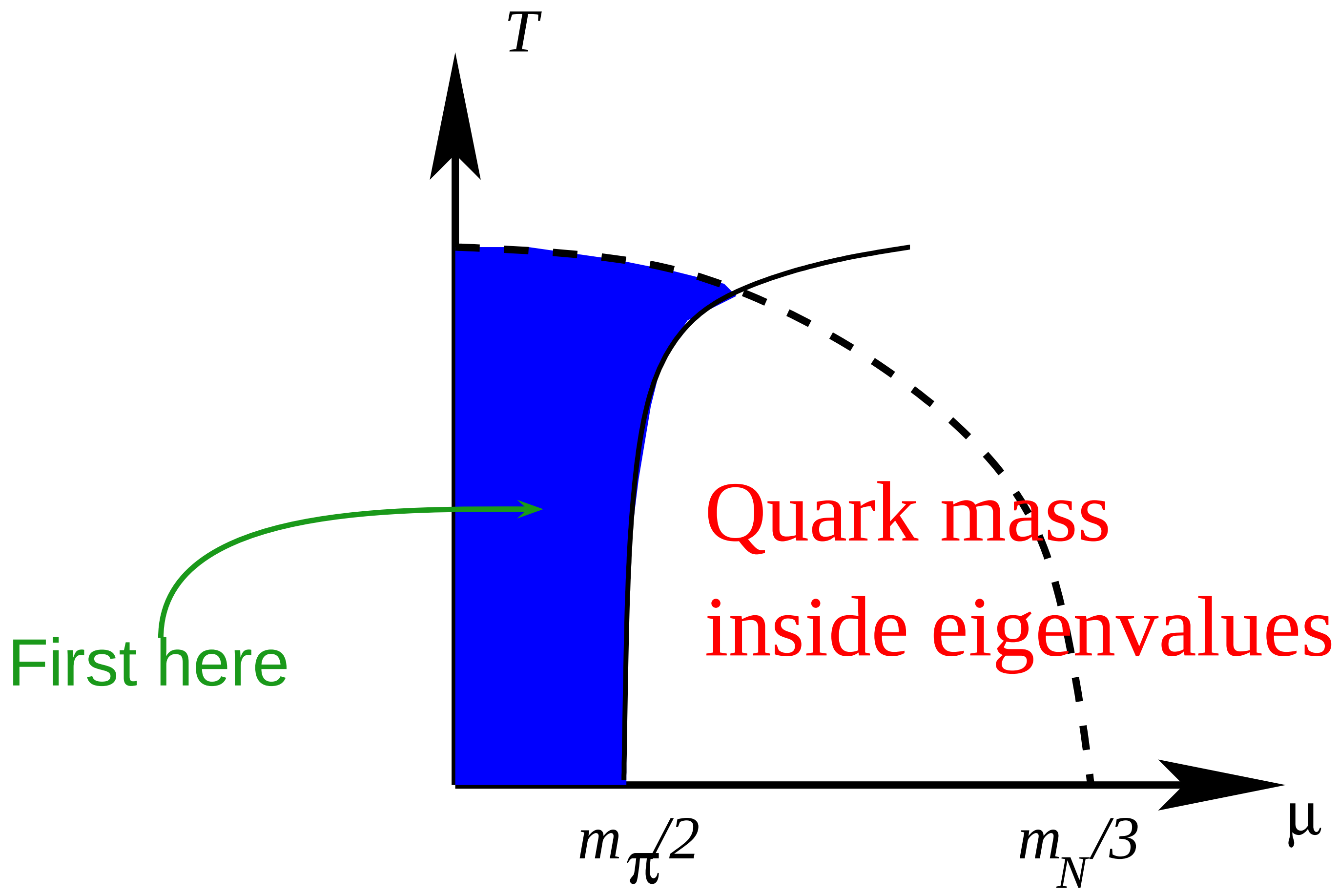
$$P_n(x,y) \equiv \langle \delta\left(x - \mathrm{Re}[n]\right)\delta\left(y - \mathrm{Im}[n]\right)\rangle$$

$\langle\delta(X)\rangle \leftrightarrow$ all $\langle X^k\rangle$

Compute all moments $\langle \mathrm{Re}[n]^k \mathrm{Im}[n]^j \rangle$ in CPT



Quark mass
inside eigenvalues

First here

The n distribution for $\mu < m_\pi/2$ $N_f = 2$

Factorization at 1-loop

$$P_n(x,y) = P_{\mathrm{Re}[n]}(x) P_{\mathrm{Im}[n]}(y)$$

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Where

$$\begin{aligned} P_{\mathrm{Re}[n]}(x) &= \frac{1}{\sqrt{\pi(\chi^B_{ud}+\chi^I_{ud})}} e^{-(x-\nu_I)^2/(\chi^B_{ud}+\chi^I_{ud})} \\ P_{\mathrm{Im}[n]}(y) &= \frac{1}{\sqrt{\pi(\chi^I_{ud}-\chi^B_{ud})}} e^{(iy+\nu_I)^2/(\chi^I_{ud}-\chi^B_{ud})} \end{aligned}$$

Note that $P_{\mathrm{Im}[n]}(y)$ takes complex values (the sign problem)

Lombardo Splittorff Verbaarschot arXiv:0910.5482

Notation

$$
\begin{aligned}
\nu_I &\equiv \frac{d}{d\mu_1}\Delta G_0(\mu_1, -\mu)\Big|_{\mu_1=\mu} \\
\chi_{ud}^{B} &\equiv \frac{d^2}{d\mu_1 d\mu_2}\Delta G_0(\mu_1, \mu_2)\Big|_{\mu_1=\mu_2=\mu} \\
\chi_{ud}^{I} &\equiv \frac{d^2}{d\mu_1 d\mu_2}\Delta G_0(-\mu_1, \mu_2)\Big|_{\mu_1=\mu_2=\mu}
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$$\Delta G_0(\mu_1,\mu_2) = V\frac{m_\pi^2 T^2}{\pi^2}\sum_{n=1}^{\infty}\frac{K_2(\frac{m_\pi n}{T})}{n^2}\left[\cosh(\frac{\mu_1-\mu_2}{T}n)-1\right]$$

The expectation value of the quark number is zero in CPT

$$\begin{aligned}
\langle n \rangle &= \int dxdy\ (x+iy)P_n(x,y) \\
&= \int dx\ xP_{\mathrm{Re}[n]}(x) + i\int dy\ yP_{\mathrm{Im}[n]}(y) \\
&= \nu_I + ii\nu_I = 0
\end{aligned}$$

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Detailed cancellation between the contribution from the real part and the imaginary part

Complex Langevin

The CL action for $y = Im[n]$ ($N_f = 2$)

$$S = -\log[P_{\mathrm{Im}[n]}(y)] = -(iy + \nu_I)^2/(\chi^I_{ud} - \chi^B_{ud})$$

Complexify $\mathrm{Im}[n]$: $y = a + ib$

Aarts personal correspondence (2009)
de Forcrand PoS (LAT2009), 10

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The flow equations for a and b decouple

$$a_{n+1} = a_n - \epsilon \frac{2a_n}{\chi_{ud}^I - \chi_{ud}^B} + \sqrt{\epsilon}\eta_n \quad \text{and} \quad b_{n+1} = b_n - \epsilon \frac{2(b_n - \nu_I)}{\chi_{ud}^I - \chi_{ud}^B}$$

a Gaussian, b no noise

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a Gaussian, b no noise CL works perfectly !

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Complex Langevin

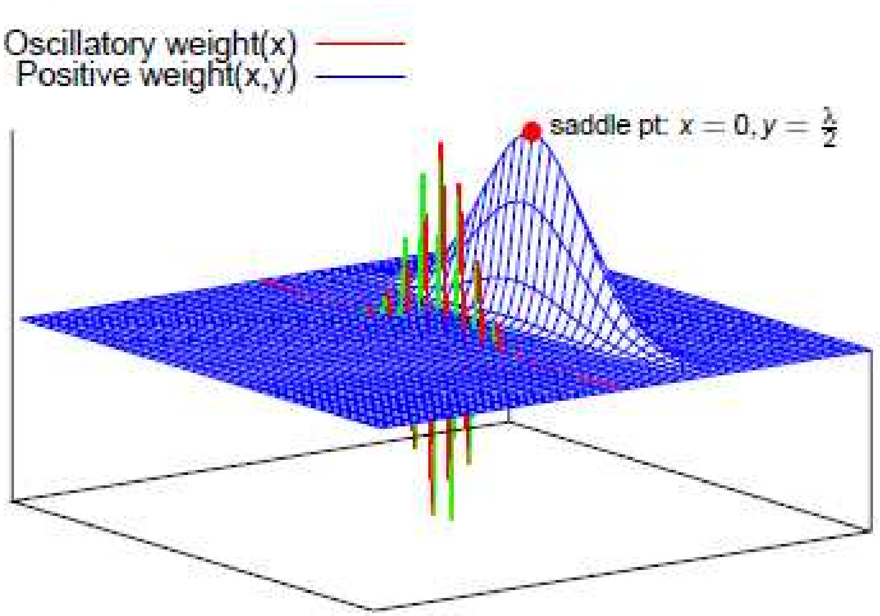

Illustration by Philippe de Forcrand

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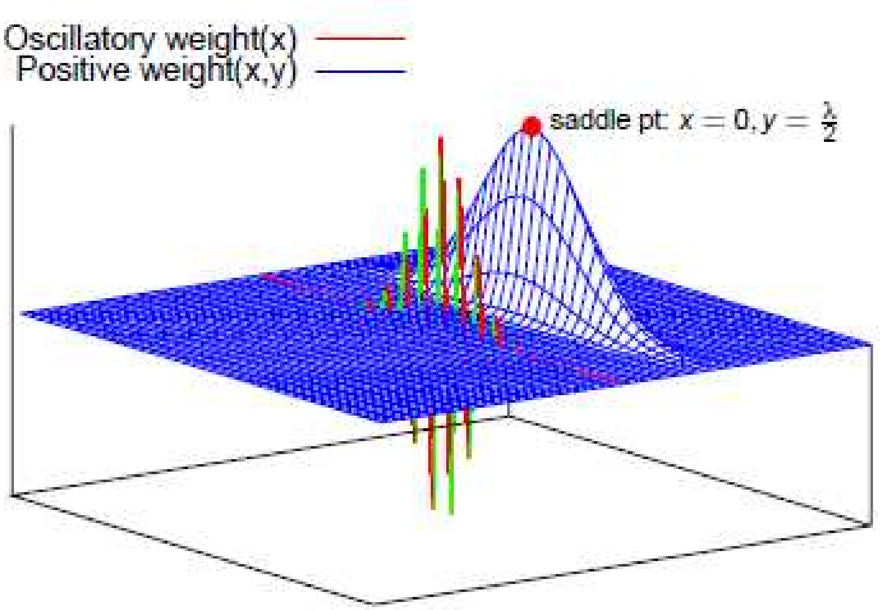

Illustration by Philippe de Forcrand

CPT tells us:

1) shift by $\mathcal{O}(V)$ in imaginary direction

2) Amplitude: real axis $\mathcal{O}(\exp(V))$; complex plane $\mathcal{O}(1)$

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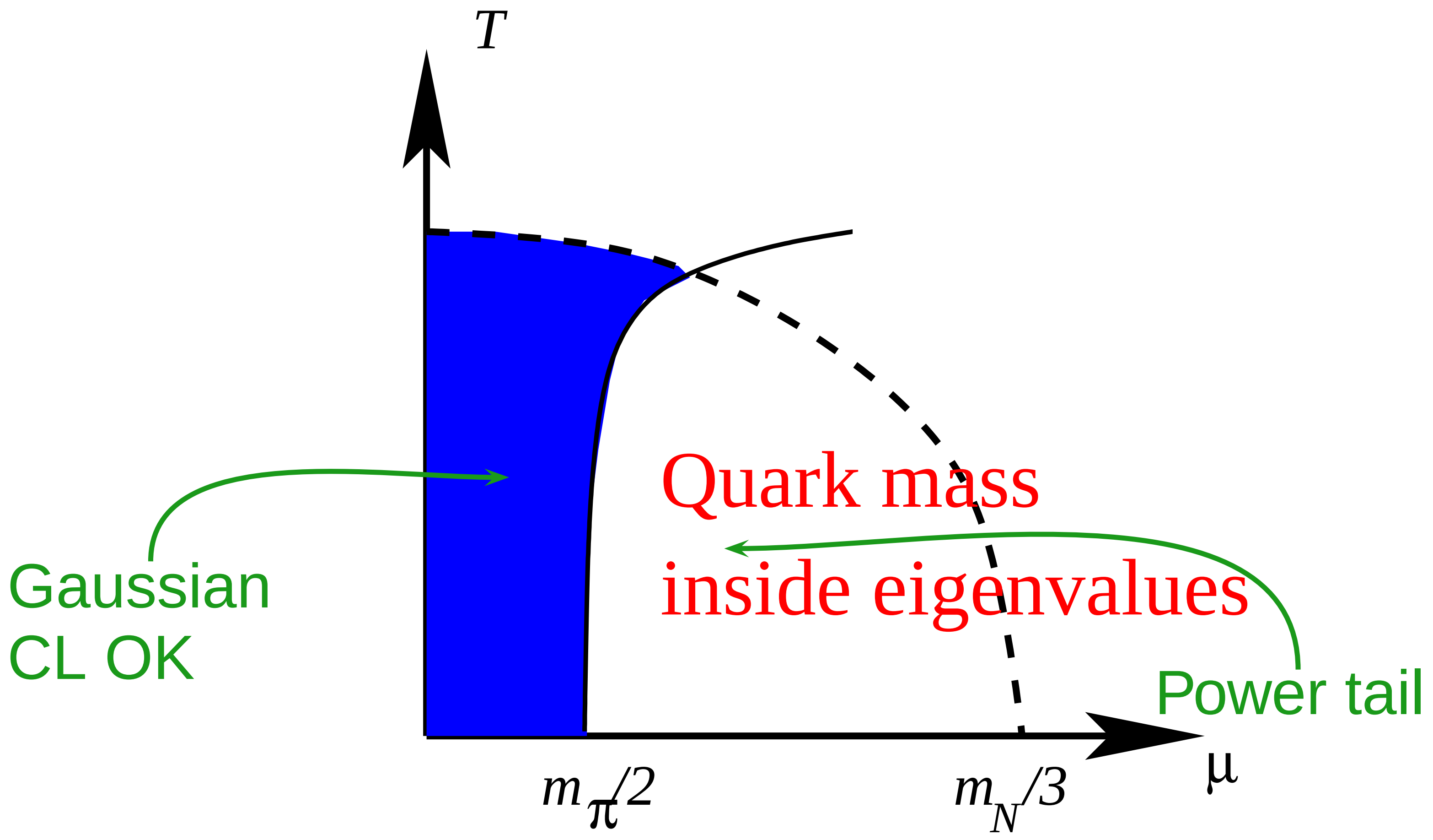
n-distribution for $\mu > m_\pi/2$



Quark mass inside eigenvalues

Gaussian CL OK

Power tail

Conclusions

Interplay between lattice QCD and analytic QCD is essential to understand QCD at $\mu \neq 0$

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Here:

Derived the distribution of n

Studied how $\langle n \rangle$ becomes zero (cancellations)

Directly linked Complex Langevin

Additional slides

The sign problem

$$Z_{1+1} = \int dA \, \det{}^2(\underset{\text{Anti Hermitian}}{D} + \underset{\text{Hermitian}}{\mu\gamma_0} + m) \, e^{-S_{\rm YM}}$$

$$\det{}^2(D + \mu\gamma_0 + m) = |\det(D + \mu\gamma_0 + m)|^2 e^{2i\theta}$$

The measure is not real and positive

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$$\det{}^2(D + \mu\gamma_0 + m) = |\det(D + \mu\gamma_0 + m)|^2 e^{2i\theta}$$

The measure is not real and positive

No Monte Carlo sampling of A_η *at* $\mu \neq 0$

In terms of the eigenvalues, z_k, of $\gamma_0(D+m)$

$$\begin{aligned} n_q &= n = \sum_k \frac{1}{z_k+\mu} \\ n_q^2 &= \sum_{k\neq l} \frac{1}{z_k+\mu}\frac{1}{z_l+\mu} \\ n^2 &= \sum_{k,l} \frac{1}{z_k+\mu}\frac{1}{z_l+\mu} = [\sum_k \frac{1}{z_k+\mu}]^2 \end{aligned}$$

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$\langle n_q^2 \rangle$ not the average of a square

not the second moment of a distribution

How large should y_{max} be in order that

$$\int_{-y_{max}}^{y_{max}} dy\ iyP_{\mathrm{Im}[n]}^{1+1}(y) \sim -\nu_I$$

The answer is:

$$y_{max} \sim \nu_I \sim V$$

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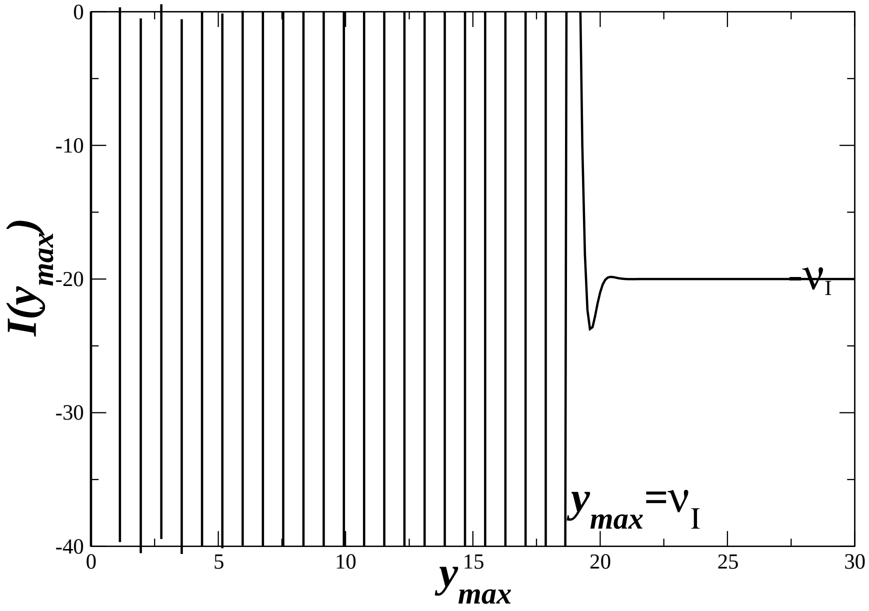
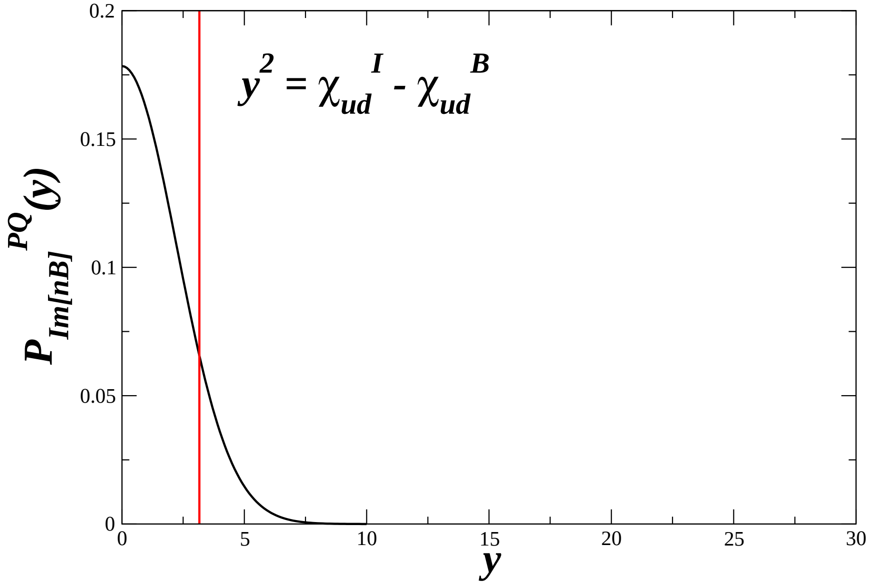
Observation: We must integrate over V periods of the oscillations in order to obtain the density

The range needed and the $\mathrm{Im}[n]$ generated ($\mu < m_\pi/2$)





$$\int_{-y_{max}}^{y_{max}} dy \; iy P^{1+1}_{\mathrm{Im}[n]}(y) \sim -\nu_I$$