Reorganizing the QCD pressure at intermediate coupling

Michael Strickland

Gettysburg College and Frankfurt Institute for Advanced Studies (FIAS)

Collaborators: Nan Su (FIAS) and Jens O. Andersen (NTNU)

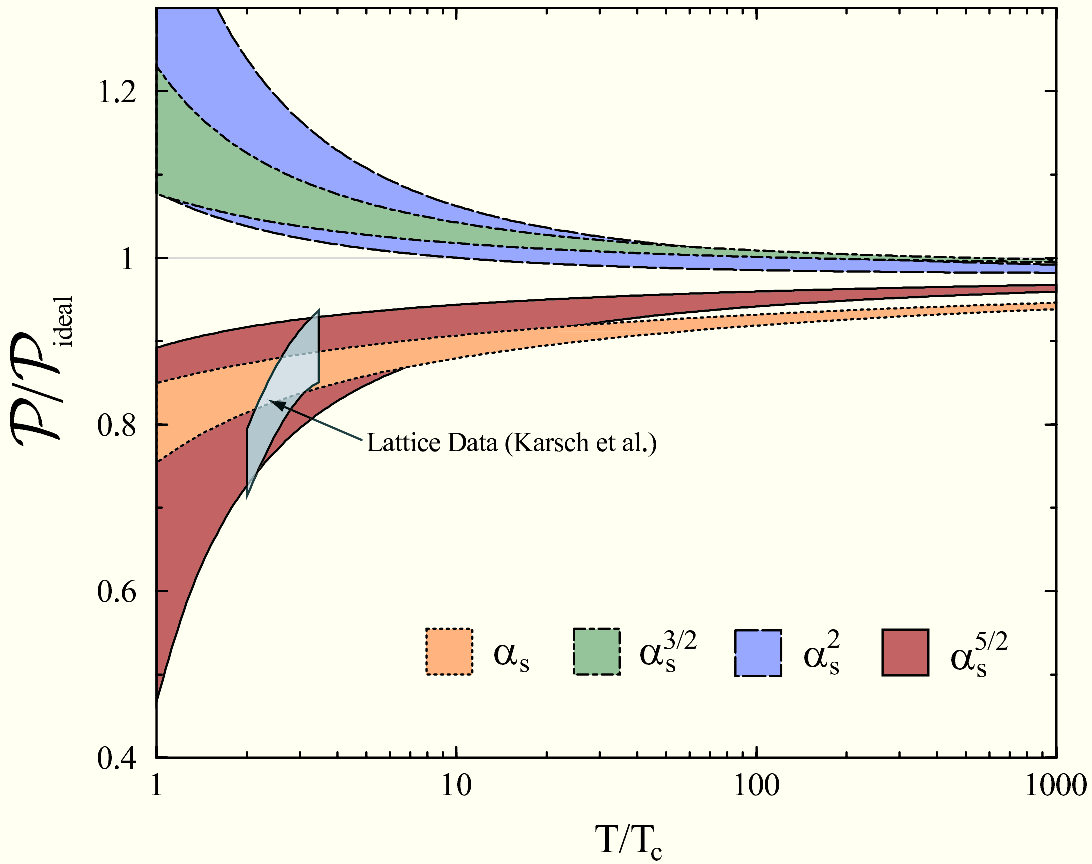
References: arXiv:0911.0676 and arXiv:0906.2936

Norwegian Winter Workshop on QCD in Extreme Conditions
Trondheim, 24-26 February 2010

Introduction - Heavy Ion Collisions $\rightarrow$ QGP or QGL?

- RHIC has made extensive studies of the matter generated during heavy ion collisions, $T_0 \sim$ 400 MeV $\sim 2\,T_c$.
- LHC will continue this investigation at even higher temperatures, $T_0 \sim$ 800-1000 MeV $\sim 4-5\,T_c$.
- Early RHIC data hinted that the perturbative approach was insufficient to explain observations, and that a strongly-coupled nearly perfect liquid may be more appropriate. LHC?
- Should we be surprised since at RHIC and LHC the running coupling expected is $g_s \sim 2$ or $\alpha_s \sim 0.3$?
- Strong coupling limit has some very nice features, but $g_s \ll \infty$.
- Can perturbative QCD results reproduce lattice data thermodynamic functions at such "intermediate" couplings ($g_s \sim 2$)?

Introduction - Perturbative QCD Thermodynamics



Perturbative QCD free energy vs temperature. $(\pi T \le \mu \le 4\pi T)$
QCD with $N_c = 3$ and $N_f = 2$.
4-d lattice results from Karsch et al, 03.

(Here $\alpha_s = g_s^2/4\pi$)

- The weak-coupling expansion of the QCD free energy, $\mathcal{F}$, has been calculated to order $\alpha_s^3 \log \alpha_s$. [1,2,3,4]

- At temperatures expected at RHIC energies, $T \sim 0.3$ GeV, the running coupling constant $\alpha_s(2\pi T)$ is approximately 1/3, or $g_s \sim 2$.

- The successive terms contributing to $\mathcal{F}$ can strictly only form a decreasing series if $\alpha_s \lesssim 1/20$ which corresponds to $T \sim 10^5$ GeV.

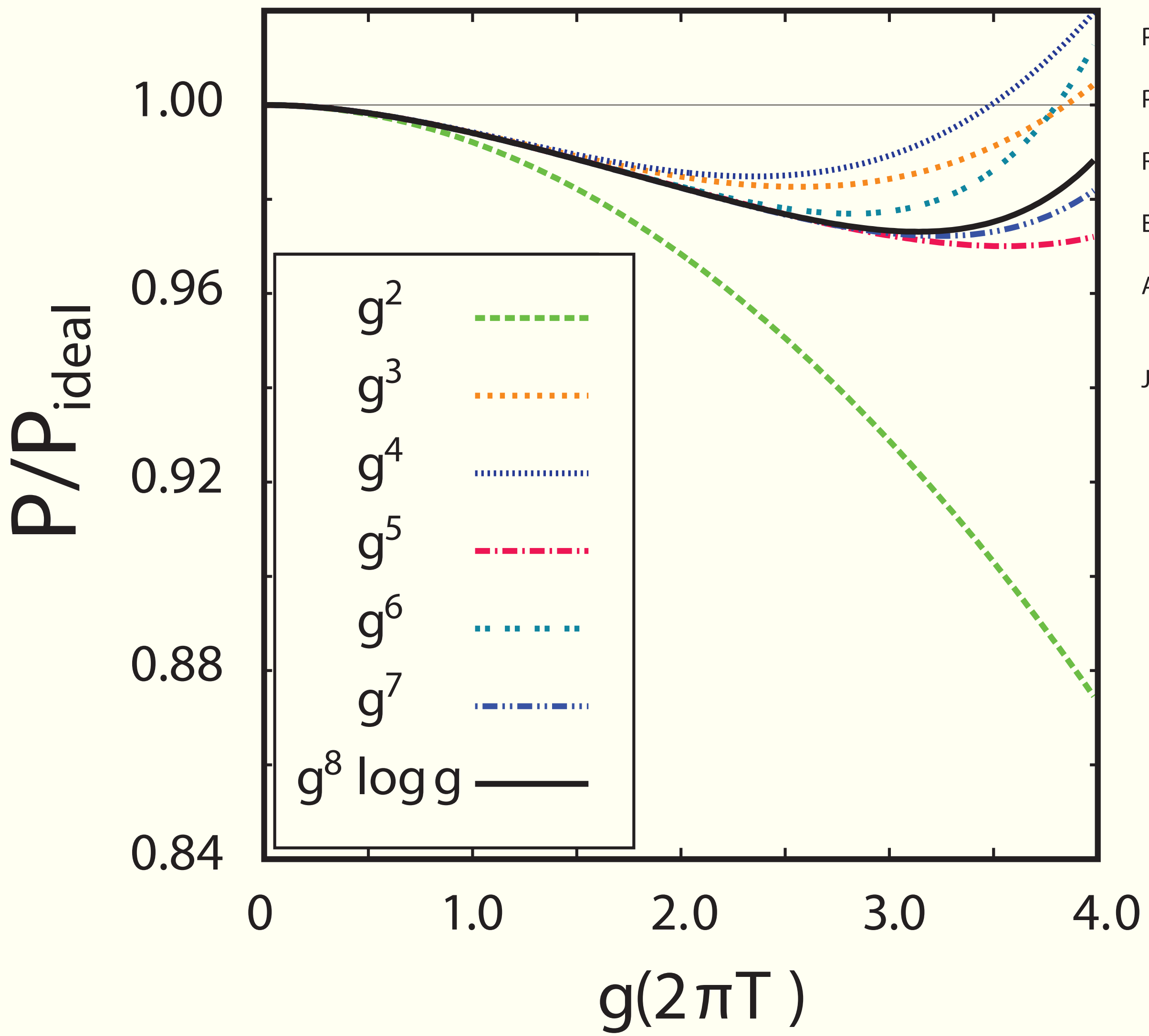
[1] Arnold and Zhai, 94/95.

[2] Kastening and Zhai, 95.

[3] Braaten and Nieto, 96.

[4] Kajantie, Laine, Rummukainen and Schröder, 02.

Introduction - Perturbative Scalar ϕ^4 Pressure



P. Arnold and C. Zhai, Phys. Rev. D50, 7603 (1994).

P. Arnold and C. Zhai, Phys. Rev. D51, 1906 (1995).

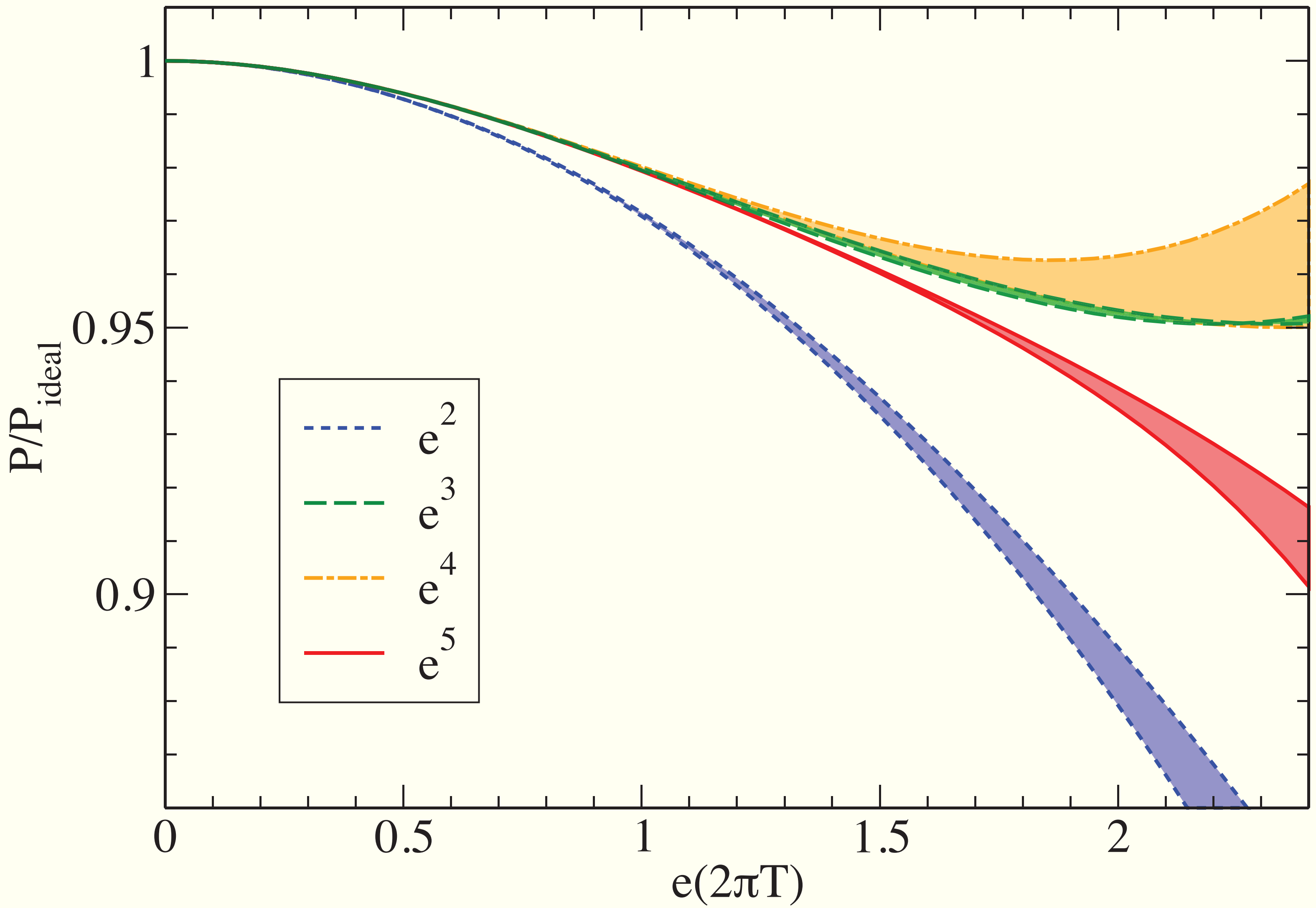
R.R. Parwani and H. Singh, Phys. Rev. D51, 4518 (1995).

E. Braaten and A. Nieto, Phys. Rev. D51, 6990 (1995).

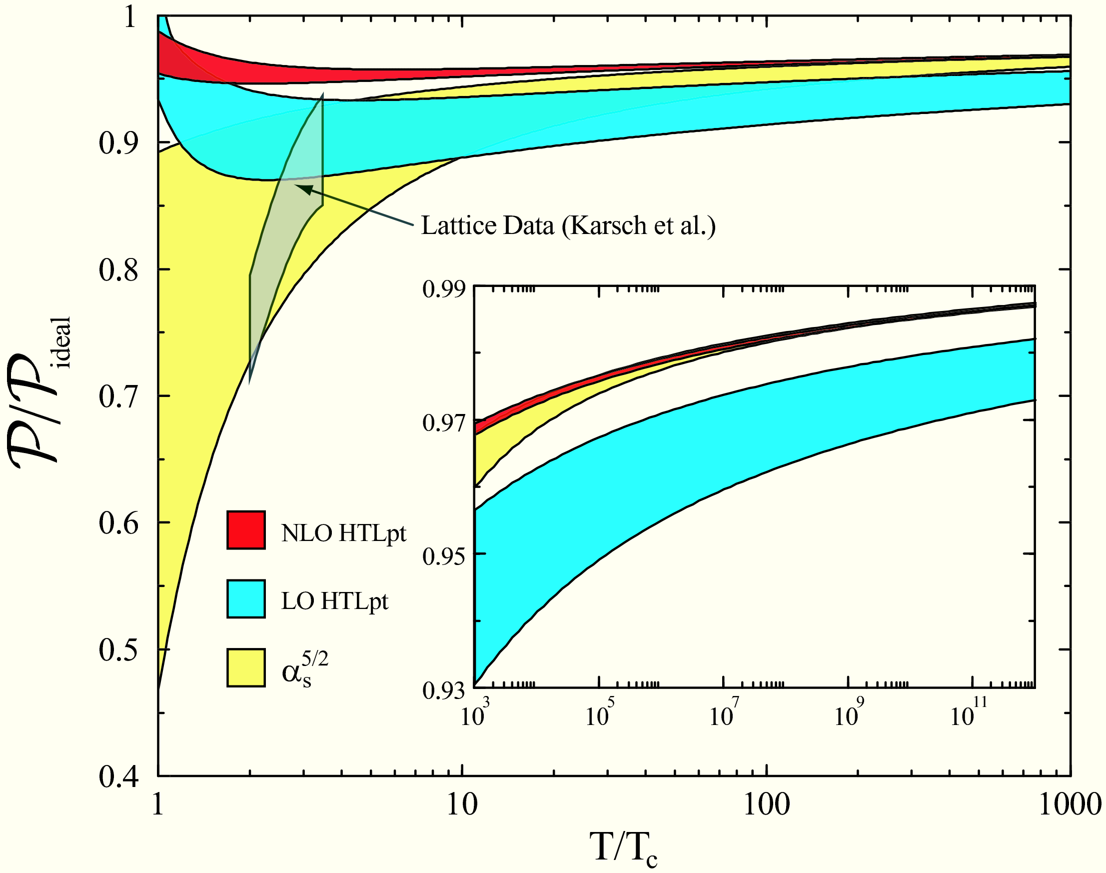
A. Gynther, M. Laine, Y. Schroder, C. Torrero, and A. Vuorinen, JHEP 04 (2007) 094

J.O. Andersen, L. Kyllingstad and L.E. Leganger, JHEP {\bf 0908}, 066 (2009).

Introduction - Perturbative QED Pressure



Introduction - NLO HTLpt Result



LO and NLO HTLpt free energy of QCD with $N_c = 3$ and $N_f = 2$ together with the perturbative prediction accurate to g^5.

- Hard-thermal-loop (HTL) perturbation theory [4,5] is a systematic, self-consistent and gauge-invariant reorganization of thermal quantum fields.
- Theory is reorganized in the sense that instead of expanding about a bare theory consisting of massless degrees of freedom, the bare theory is one with quasiparticle effects built in.
- Hard-thermal-loop perturbation theory is formulated in Minkowski space, therefore it is in principle possible to carry out real time calculations.
- Interested in $T > 2 - 3\,T_c$.

[4] Andersen, Braaten, Strickland, 99/99/99.

[5] Andersen, Braaten, Petitgirard, Strickland, 02; Andersen, Petitgirard, Strickland, 03.

But there is still work to do!

- Problems remain:
  - g^4 and g^5 terms can't be fully fixed at NLO.
  - For example, when the NLO HTLpt is expanded in a truncated series in g, it is found that the g^5 term has approximately the right magnitude, but the **wrong sign** when comparing to the known weak-coupling expansion.
  - Running coupling doesn't enter at NLO. At this order, running coupling needs to be put in by hand.
- Can be fixed by going to NNLO.

Time to roll up your sleeves . . .

Anharmonic Oscillator

- Consider the perturbation series for the ground state energy, E, of a simple anharmonic oscillator with potential

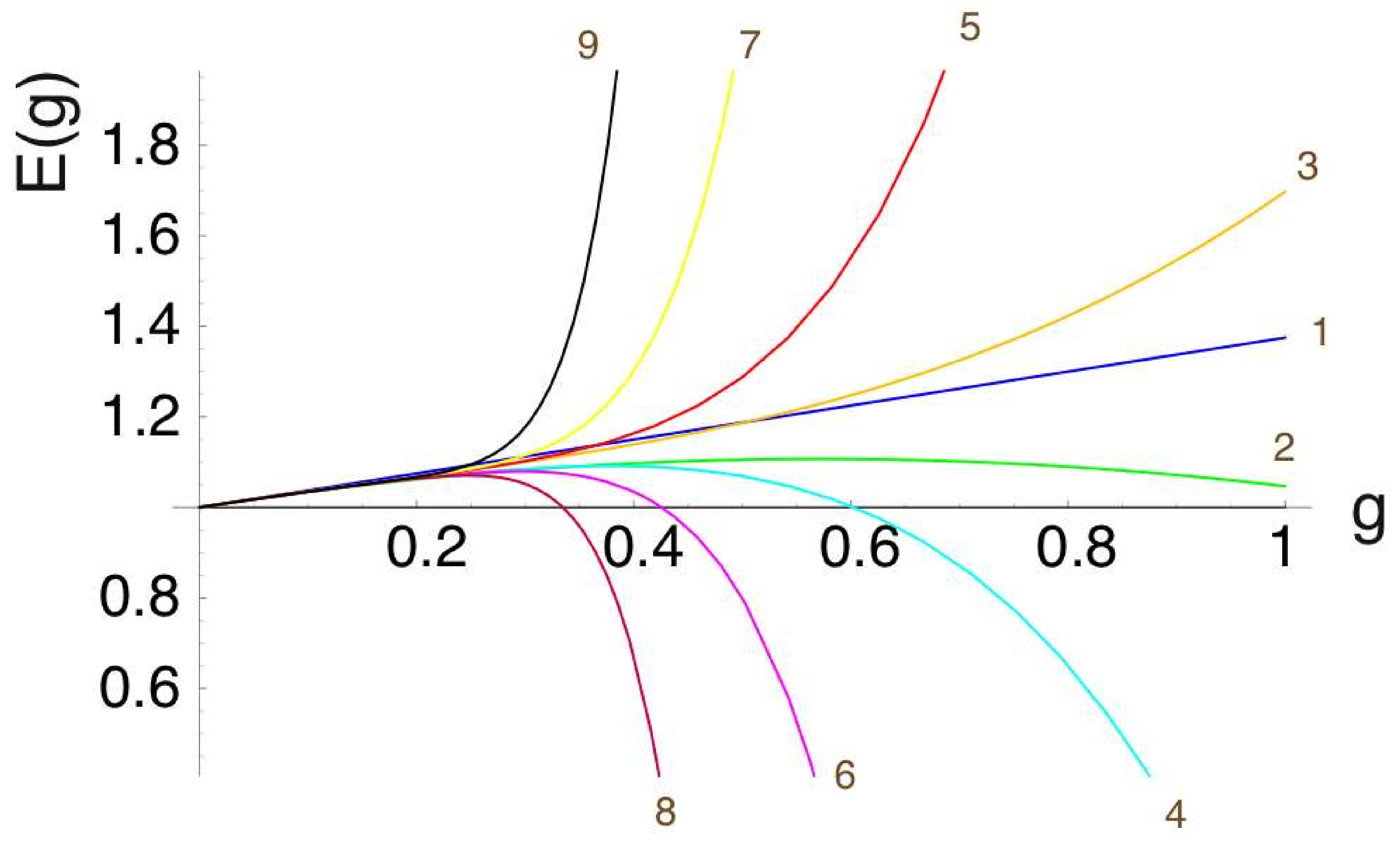
$$V(x) = \frac{1}{2}\omega^2 x^2 + \frac{g}{4}x^4 \qquad (\omega^2, g > 0)$$

- Weak-coupling expansion of the ground state energy $E(g)$ is known to all orders (Bender and Wu 69/73)

$$E(g) = \omega \sum_{n=0}^{\infty} c_n^{\mathrm{BW}} \left(\frac{g}{4\omega^3}\right)^n, \quad c_n^{\mathrm{BW}} = \left\{\frac{1}{2}, \frac{3}{4}, -\frac{21}{8}, \frac{333}{16}, -\frac{30885}{16}, \ldots\right\}$$

- $\lim_{n\to\infty} c_n^{\mathrm{BW}} = (-1)^{n+1}\sqrt{\frac{6}{\pi^3}}3^n(n-\frac{1}{2})!$
- Because of the factorial growth, the expansion is an asymptotic series with zero radius of convergence!

Anharmonic Oscillator



Variational Perturbation Theory (Janke and Kleinert 95/97)

- Split the harmonic term into two pieces and treat the second as part of the interaction

$$\omega^2 \to \Omega^2 + \left(\omega^2 - \Omega^2\right) \quad \Longrightarrow \quad E_N(g,r) = \Omega \sum_{n=0}^{N} c_n(r) \left(\frac{g}{4\Omega^3}\right)^n$$

where $r \equiv \frac{2}{g}\left(\omega^2 - \Omega^2\right)$

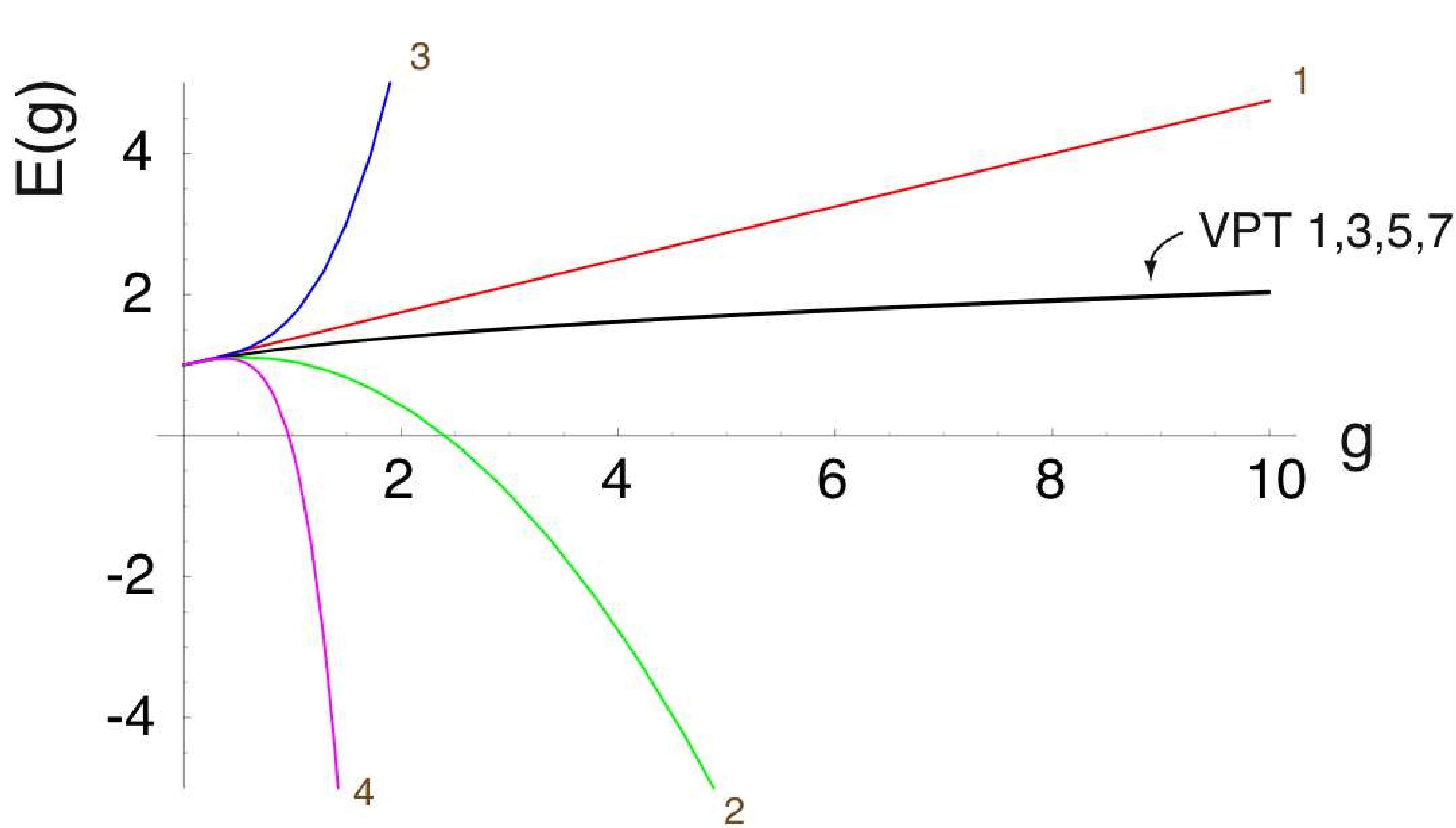
- The new coefficients c_n can be obtained by

$$c_n(r) = \sum_{j=0}^{n} c_j^{\mathrm{BW}} \begin{pmatrix} (1-3j)/2 \\ n-j \end{pmatrix} (2r\Omega)^{n-j}$$

- Fix Ω_N by requiring that at each order N

$$\left.\frac{\partial E_N}{\partial \Omega}\right|_{\Omega=\Omega_N} = 0$$

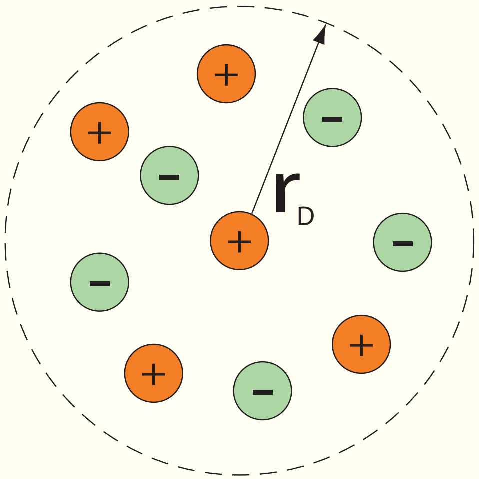
Variational Perturbation Theory



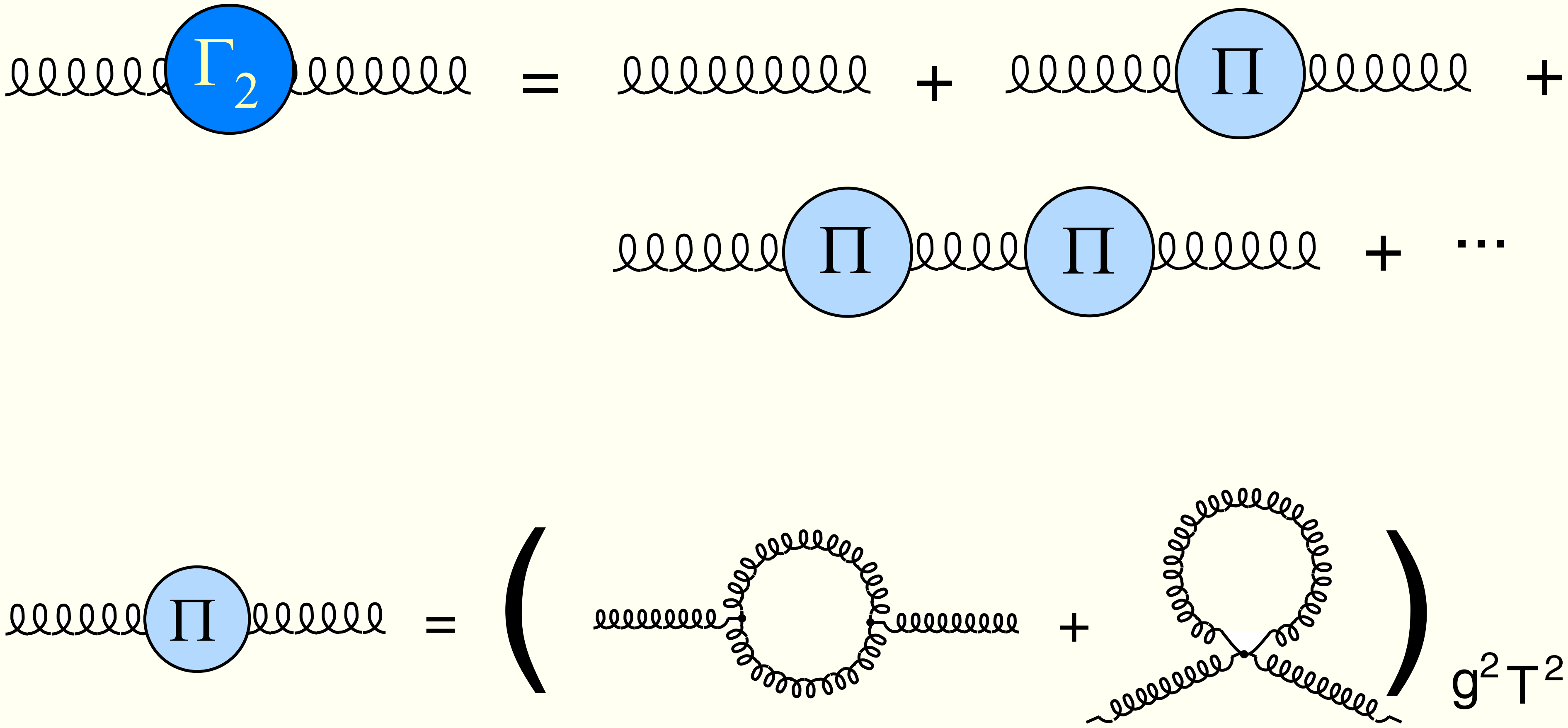
Finite Temperature QED/QCD Primer

- Long-wavelength chromoelectric fields with momentum $k \sim \lambda^{-1} \sim gT$ are “screened” by an induced mass called the Debye mass m_D.
- At high temperatures particles $\rightarrow$ massive quasiparticles
- $k \sim gT$ defines the *soft scale*, $k \sim T$ defines the *hard scale*.
- The inverse Debye mass is called the “Debye screening length”, ie $r_D = 1/m_D$.

$$V_{\text{Coloumb}} \rightarrow V_{\text{Debye}} \sim \frac{e^{-m_D r}}{r} \sim \frac{e^{-r/r_D}}{r}$$



Hard Thermal Loops: Propagator Resummation



Finite Temperature QED/QCD Primer

- At leading order in the coupling constant $m_D^2 = g^2 T^2$ for SU(3) Yang-Mills and $m_D^2 = e^2 T^2/3$ for QED; however, this is not the end of the story.
- Since QCD and QED are gauge theories, there are relationships between the n-point functions which must be maintained in order to preserve gauge invariance.
- These are called Ward-Takahashi or Slavnov-Taylor identities, e.g. $p_\mu \Gamma^\mu(p,q,r) = S^{-1}(q) - S^{-1}(r)$ must be obeyed by the fermion-gauge field vertex function Γ^μ and propagator S.
- All n-point functions of the theory must related (Braaten and Pisarski 92)

$$\longrightarrow \quad \mathcal{L}_{\mathrm{HTL}} = -\frac{1}{2} m_D^2 \,\mathrm{Tr}\left(G_{\mu\alpha} \left\langle \frac{y^\alpha y^\beta}{(y \cdot D)^2} \right\rangle_y G^\mu{}_\beta \right)$$

The Hard Thermal Loop Effective Action

$$\mathcal{L}_{\mathrm{YM}} + \mathcal{L}_{\mathrm{HTL}} = \frac{1}{4} G^{\mu\nu} G_{\mu\nu} - \frac{1}{2} m_D^2 \mathrm{Tr} \left(G_{\mu\alpha} \left\langle \frac{y^\alpha y^\beta}{(y \cdot D)^2} \right\rangle_y G^\mu{}_\beta \right)$$

- $G_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + ig[A_\mu, A_\nu]$
- $D_\mu = \partial_\mu + igA_\mu$
- Expanding to quadratic order in A gives dressed propagator (2-point function)
- Expanding to cubic order in A gives dressed gluon three-vertex
- Expanding to quartic order in A gives dressed gluon four-vertex
- And so on . . . contains an infinite number of higher order vertices which all exactly satisfy the appropriate Ward-Takahashi or Slavnov-Taylor identities

Hard-Thermal-Loop Perturbation Theory (HTLpt)

Hard-thermal-loop perturbation theory is a reorganization of the perturbative series for QCD

$$\mathcal{L}_{\rm HTLpt} = \left(\mathcal{L}_{\rm QCD} + \mathcal{L}_{\rm HTL}\right)\Big|_{g\to\sqrt{\delta}g} + \Delta\mathcal{L}_{\rm HTL}(g, m_D^2(1-\delta))$$

The HTL "improvement" term is

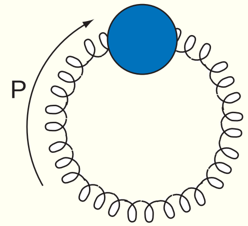
$$\mathcal{L}_{\rm HTL} = -\frac{1}{2}(1-\delta)m_D^2 \,{\rm Tr}\left(G_{\mu\alpha}\left\langle \frac{y^\alpha y^\beta}{(y\cdot D)^2}\right\rangle_y G^\mu{}_\beta\right)$$

where $\langle\cdots\rangle_y$ indicates an angle average. Formally we expand in a power series about $\delta = 0$ and set $\delta = 1$ at the end. The parameter δ provides a method for formal "bookkeeping" of what is in the bare action and what is considered an interaction.

HTLpt: 1-loop free energy, mass expansion

- Separation into hard and soft contributions ($d = 3 - 2\epsilon$)

$$\mathcal{F}_g = -\frac{1}{2}\sum\!\!\!\!\!\!\!\int_P \{(d-1)\log[-\Delta_T(P)] \;+\; \log\Delta_L(P)\}$$



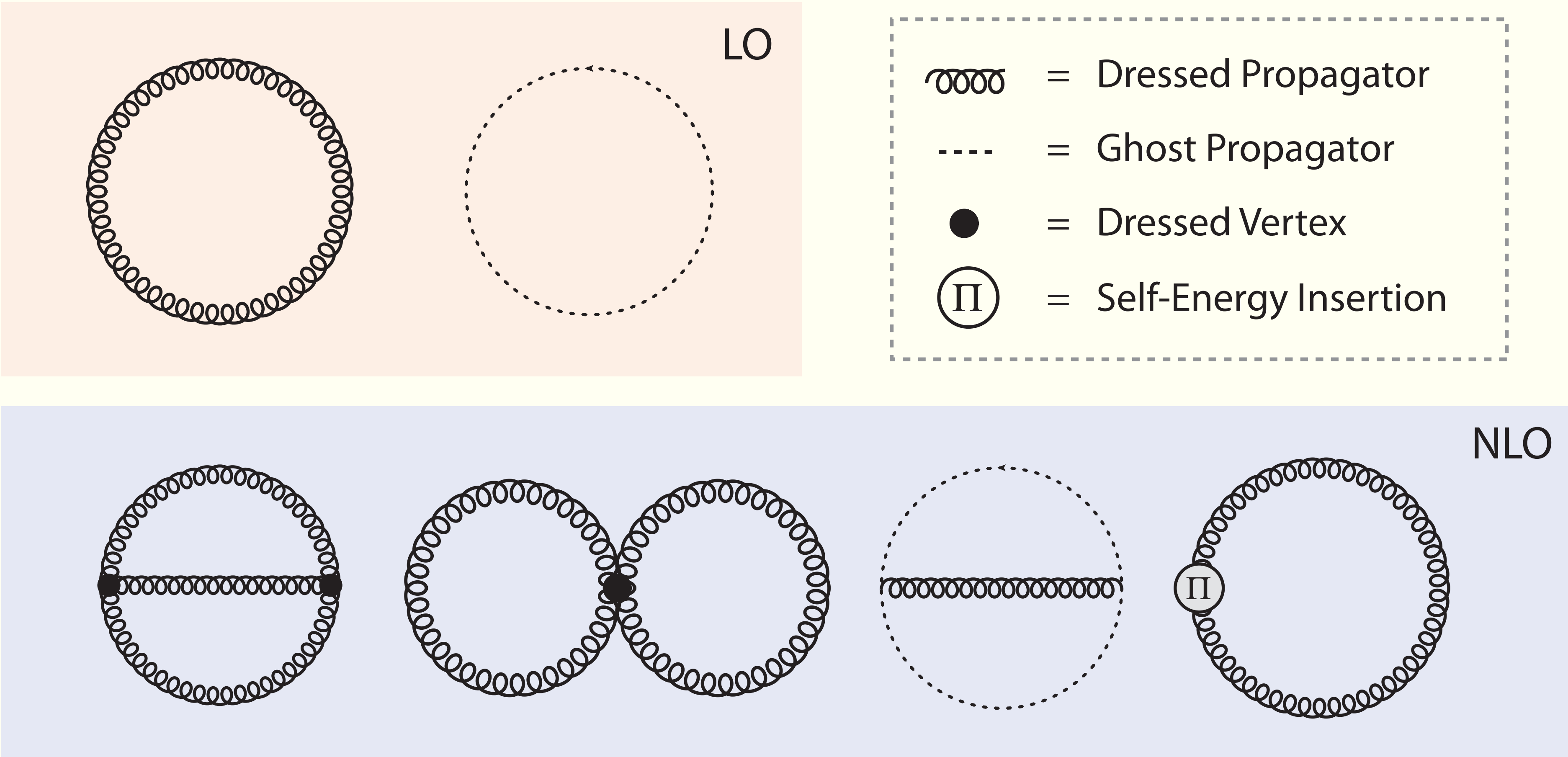
  - Hard momenta $(\omega, \mathbf{p} \sim T)$

$$\mathcal{F}_g^{(h)} \;=\; \frac{d-1}{2}\sum\!\!\!\!\!\!\!\int_P \log(P^2) + \frac{1}{2}m_D^2\sum\!\!\!\!\!\!\!\int_P \frac{1}{P^2} - \frac{1}{4(d-1)}m_D^4\sum\!\!\!\!\!\!\!\int_P \left[\frac{1}{(P^2)^2} - 2\,\frac{1}{p^2P^2} - 2d\,\frac{1}{p^4}\mathcal{T}_P + \;2\,\frac{1}{p^2P^2}\mathcal{T}_P + d\,\frac{1}{p^4}(\mathcal{T}_P)^2\right] + \mathcal{O}(m_D^6)$$

  - Soft momenta $(\omega, \mathbf{p} \sim gT)$

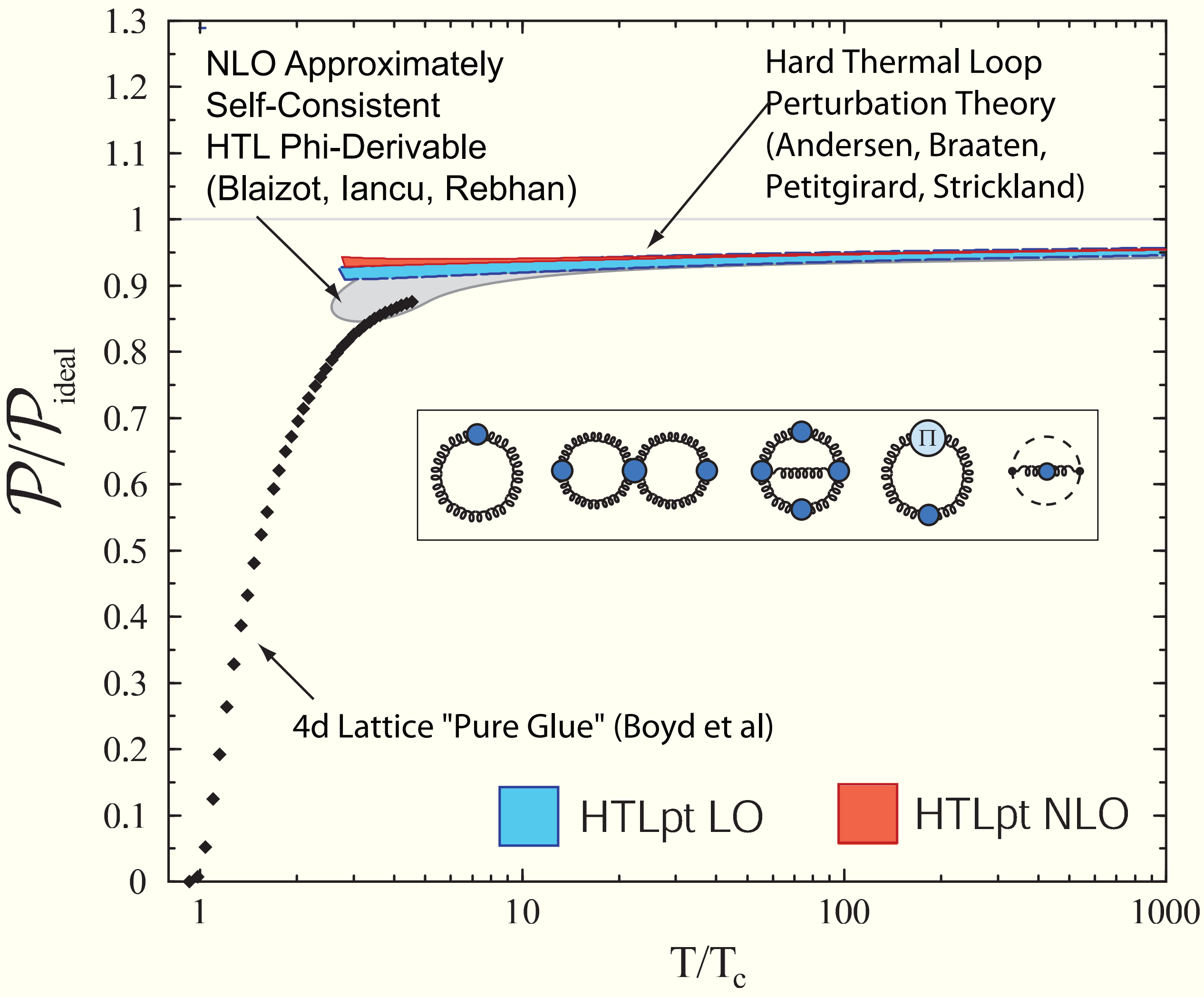
$$\mathcal{F}_g^{(s)} \;=\; \frac{1}{2}T\int_{\mathbf{p}} \log(p^2 + m_D^2)$$

HTLpt: 1- and 2-loop diagrams for QCD



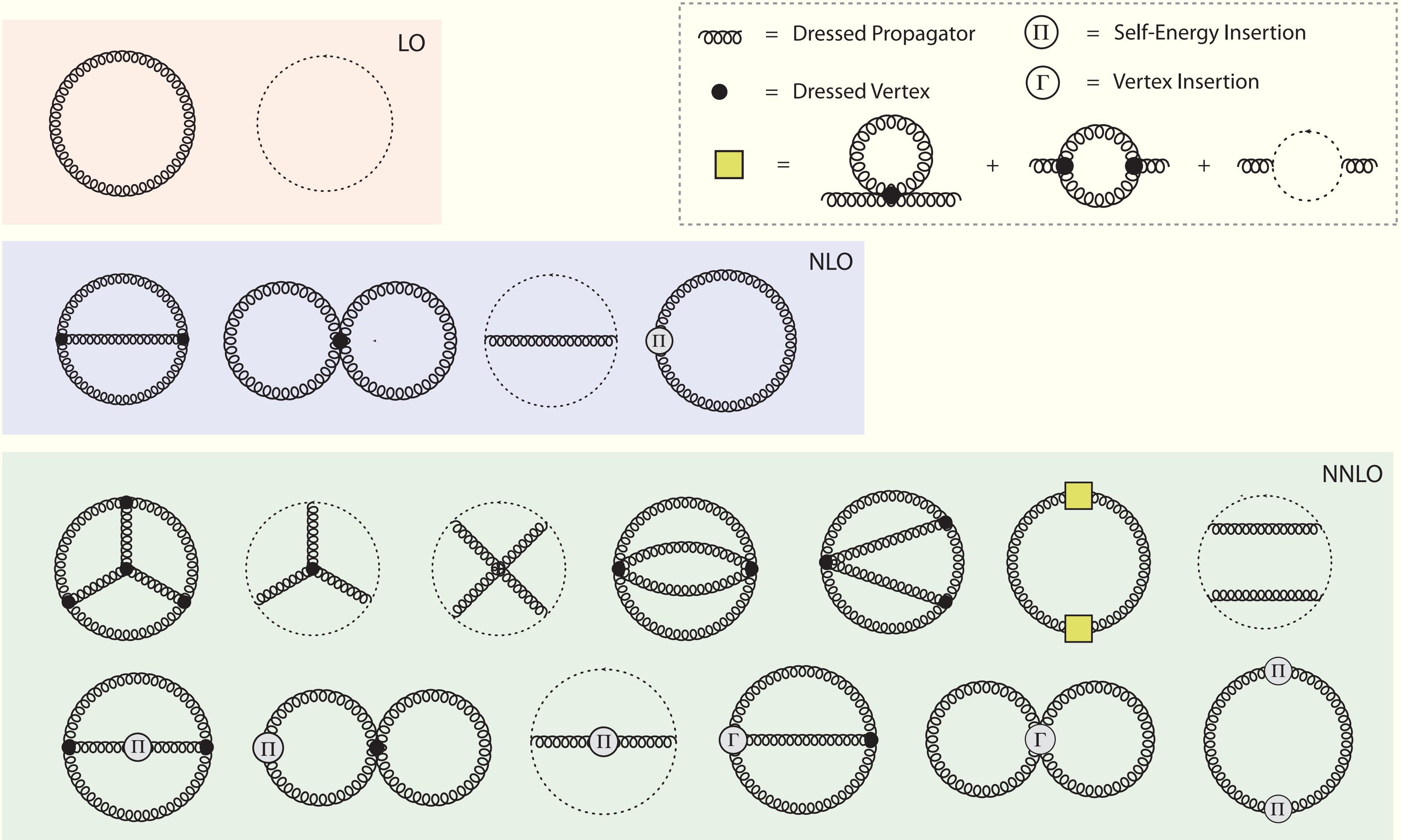
LO and NLO HTLpt Yang Mills diagrams

HTLpt: 1- and 2-loop free energy for pure glue



LO and NLO HTLpt free energy of pure glue vs temperature
Andersen, Braaten, Petitgirard, Strickland, 02.

Pure-Glue diagrams through NNLO in HTLpt



NNLO HTLpt thermodynamic functions for pure glue

- We have recently completed the NNLO calculation of the HTLpt thermodynamic potential for pure Yang-Mills theory. The result is completely analytic. Defining $\hat{x} \equiv x/(2\pi T)$ we find

$$\frac{\Omega_{\rm NNLO}}{\mathcal{F}_{\rm ideal}} = 1 - \frac{15}{4}\hat{m}_D^3 + \frac{N_c\alpha_s}{3\pi}\left[-\frac{15}{4} + \frac{45}{2}\hat{m}_D - \frac{135}{2}\hat{m}_D^2 - \frac{495}{4}\left(\log\frac{\hat{\mu}}{2} + \frac{5}{22} + \gamma\right)\hat{m}_D^3\right]$$
$$+ \left(\frac{N_c\alpha_s}{3\pi}\right)^2\left[\frac{45}{4}\frac{1}{\hat{m}_D} - \frac{165}{8}\left(\log\frac{\hat{\mu}}{2} - \frac{72}{11}\log\hat{m}_D - \frac{84}{55} - \frac{6}{11}\gamma - \frac{74}{11}\frac{\zeta'(-1)}{\zeta(-1)} + \frac{19}{11}\frac{\zeta'(-3)}{\zeta(-3)}\right)\right.$$
$$\left. + \frac{1485}{4}\left(\log\frac{\hat{\mu}}{2} - \frac{79}{44} + \gamma + \log 2 - \frac{\pi^2}{11}\right)\hat{m}_D\right]$$

- The counterterms necessary to remove all divergences are

$$\delta\Delta\alpha_s = -\frac{11N_c}{12\pi\epsilon}\alpha_s^2\delta^2 \;, +\mathcal{O}(\delta^3\alpha_s^3)$$
$$\Delta m_D^2 = \left(-\frac{11N_c}{12\pi\epsilon}\alpha_s\delta + \mathcal{O}(\delta^2\alpha_s^2)\right)(1-\delta)m_D^2 \;,$$
$$\Delta\mathcal{E}_0 = \left(\frac{N_c^2-1}{128\pi^2\epsilon} + \mathcal{O}(\delta\alpha_s)\right)(1-\delta)^2 m_D^4$$

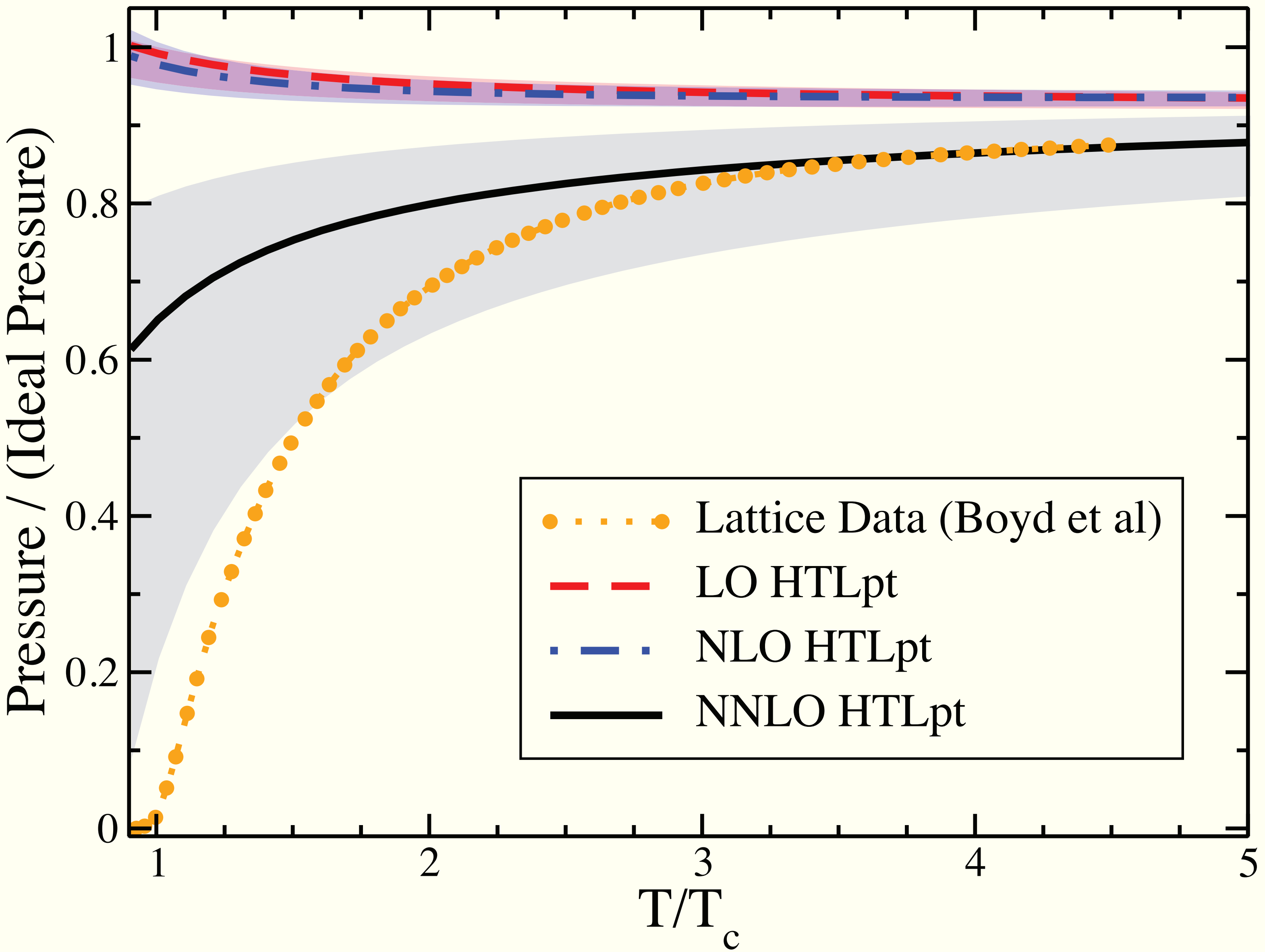
Give me the bad news first . . .

- The resulting variational equation for the Debye mass has two complex solutions ($m_D = \alpha \pm i\beta$) for all g.
- The variational solutions for the free energy are therefore also complex. The same thing happens in QED.
- We *believe* this is due to our truncation of the m_D expansion but have no conclusive proof of this.
- As a way around this problem instead for the Debye mass we use a perturbative NLO expression for hard contribution to the Debye mass which was derived using effective field theory methods by Braaten and Neito in 1995.

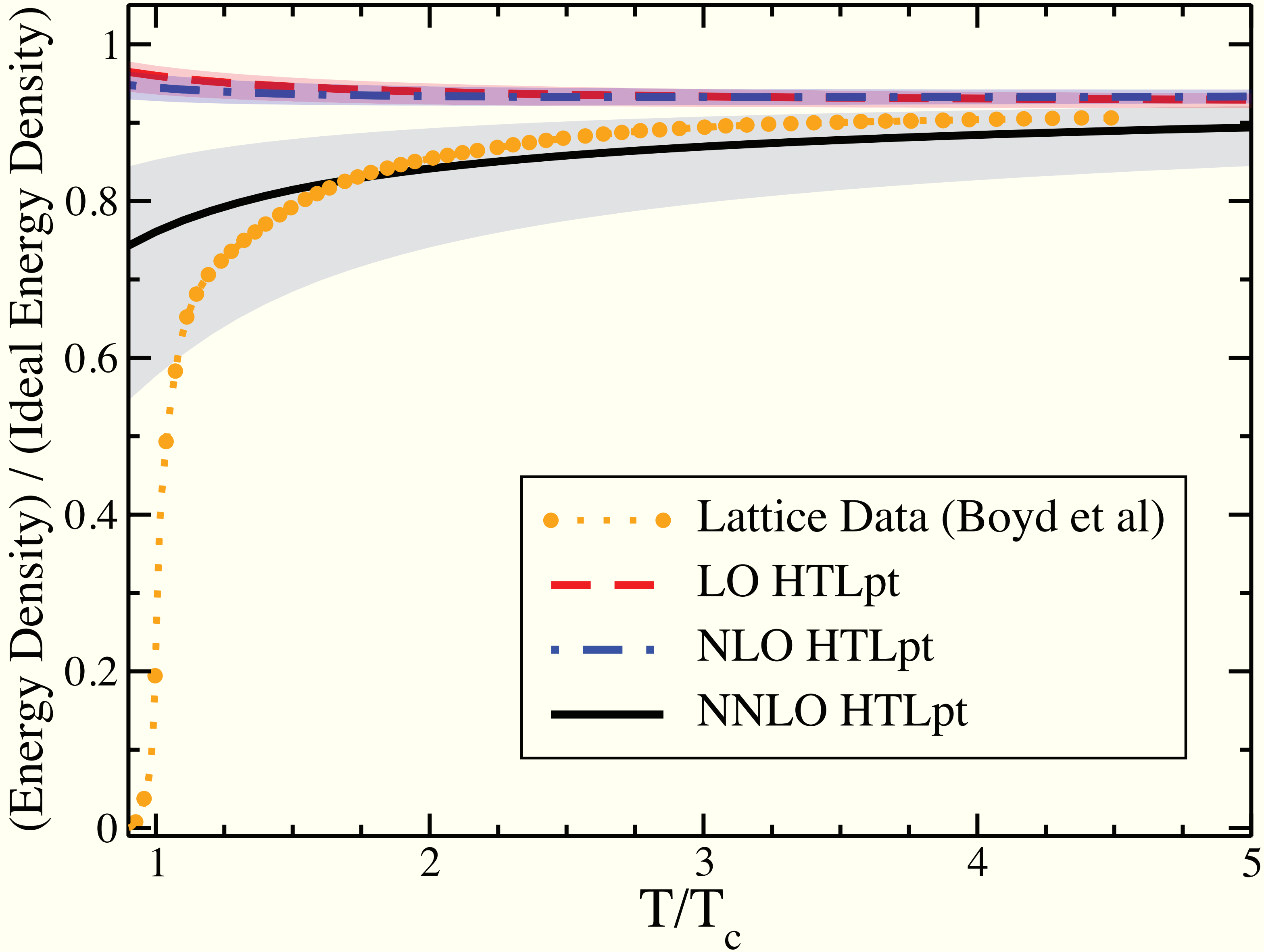
$$\frac{\hat{m}_D^2}{\hat{m}_{D,\mathrm{LO}}^2} = 1 + \frac{N_c \alpha_s}{3\pi}\left(\frac{5}{4} + \frac{11}{2}\gamma + \frac{11}{2}\log\frac{\hat{\mu}}{2}\right)$$

- For α_s we use the standard 3-loop running.

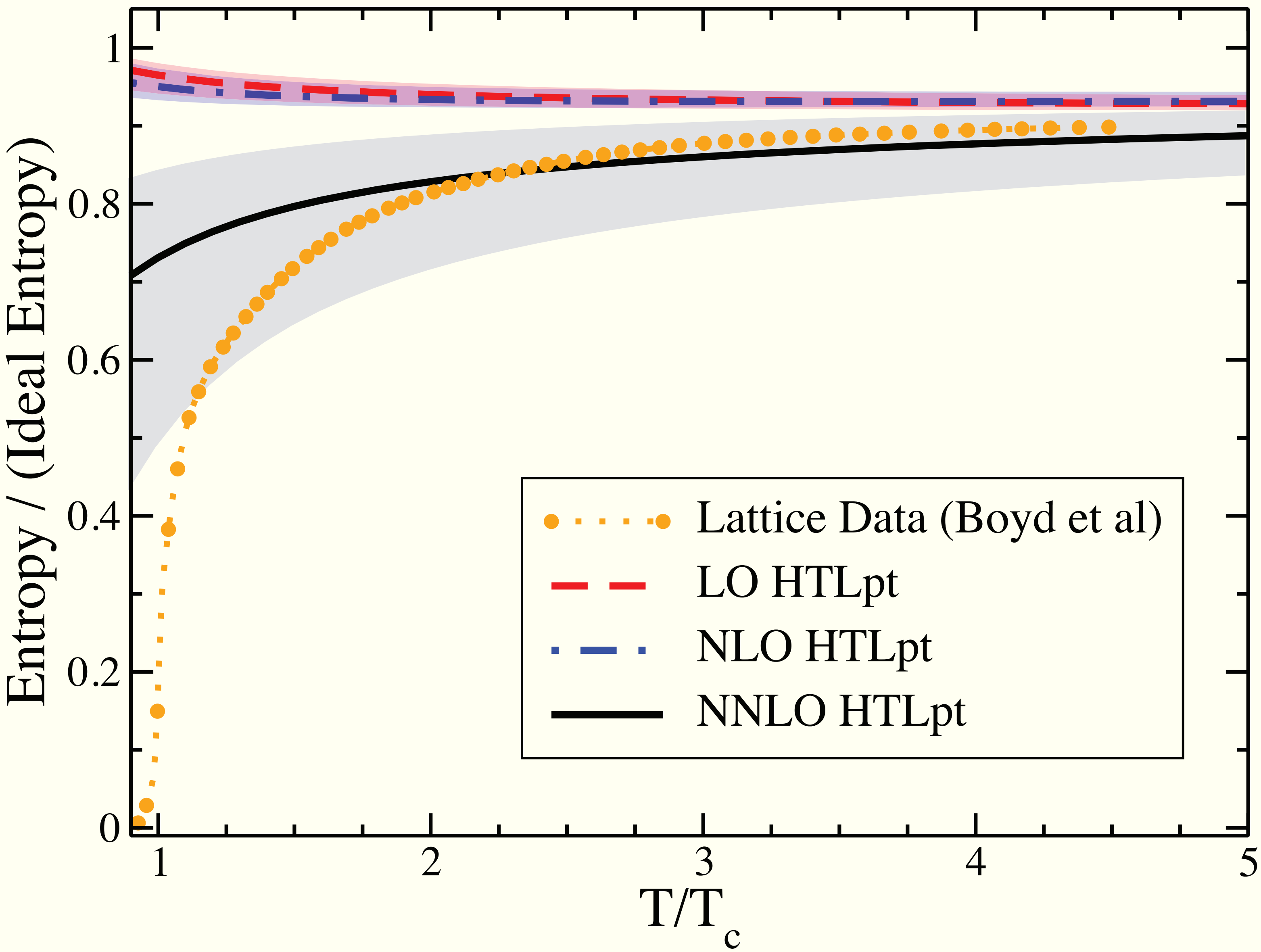
$N_c = 3$ Pure-Glue NNLO Pressure (arXiv:0911.0676)



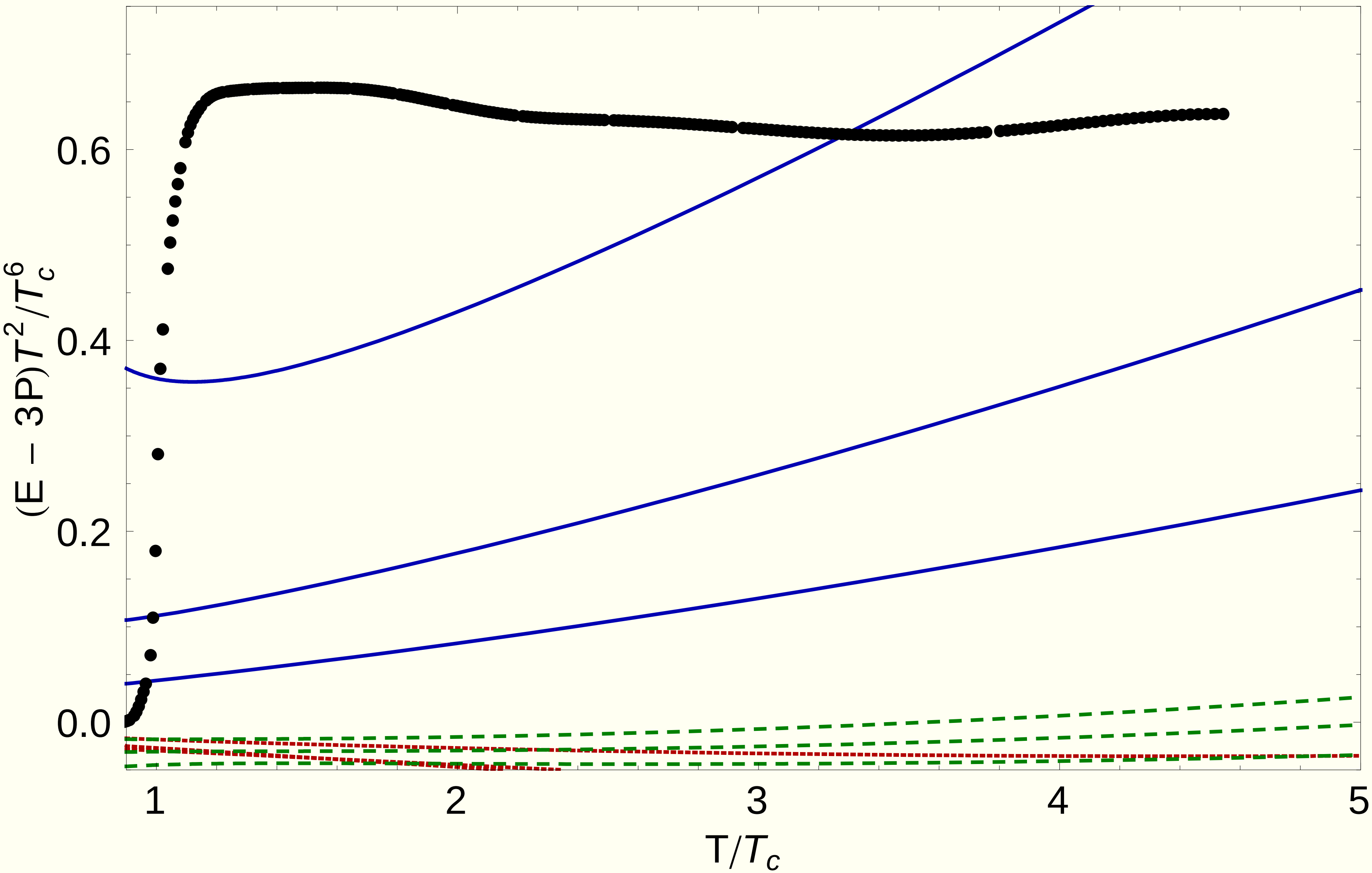
$N_c = 3$ Pure-Glue NNLO Energy (arXiv:0911.0676)



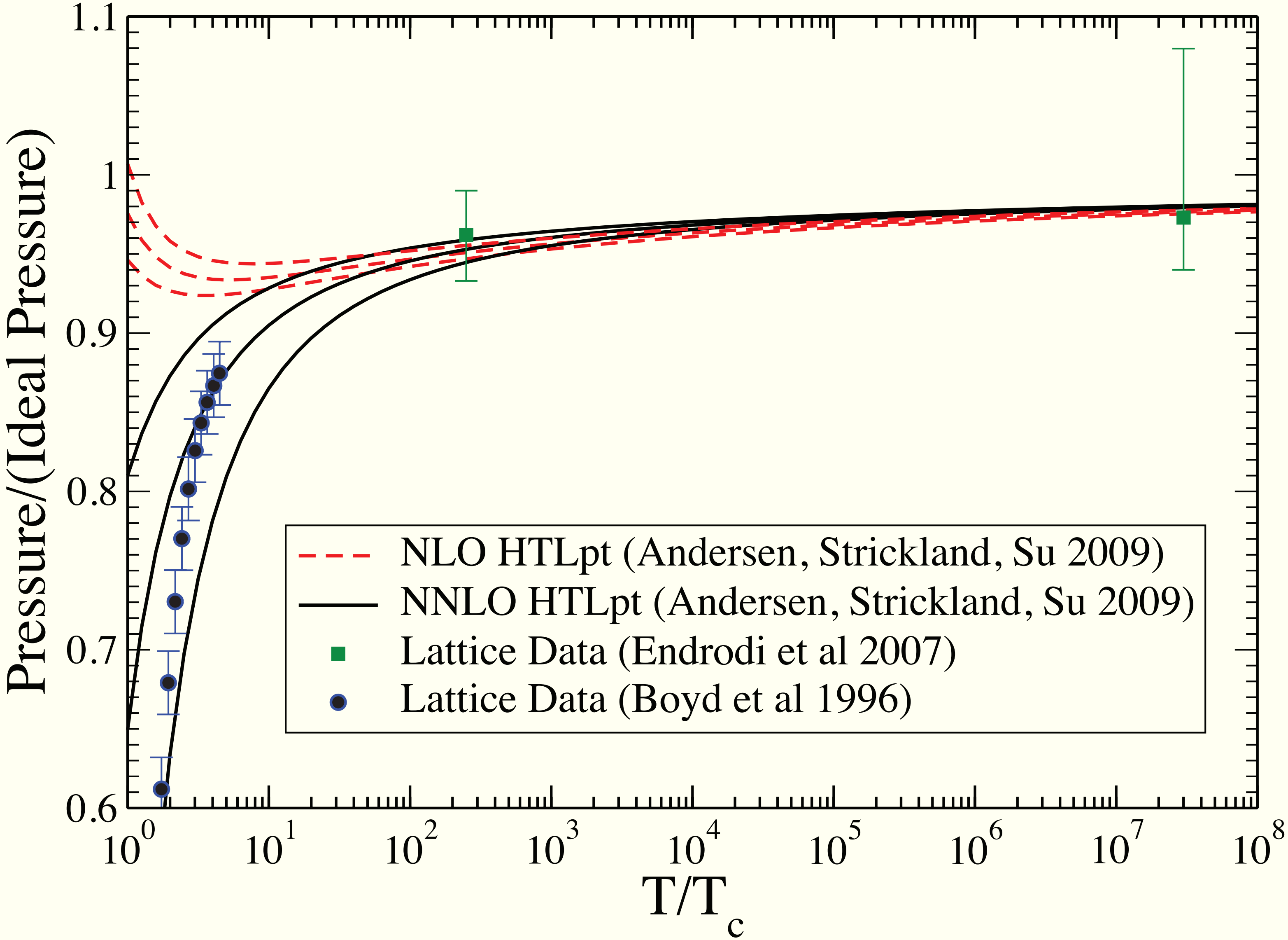
$N_c = 3$ Pure-Glue NNLO Entropy (arXiv:0911.0676)



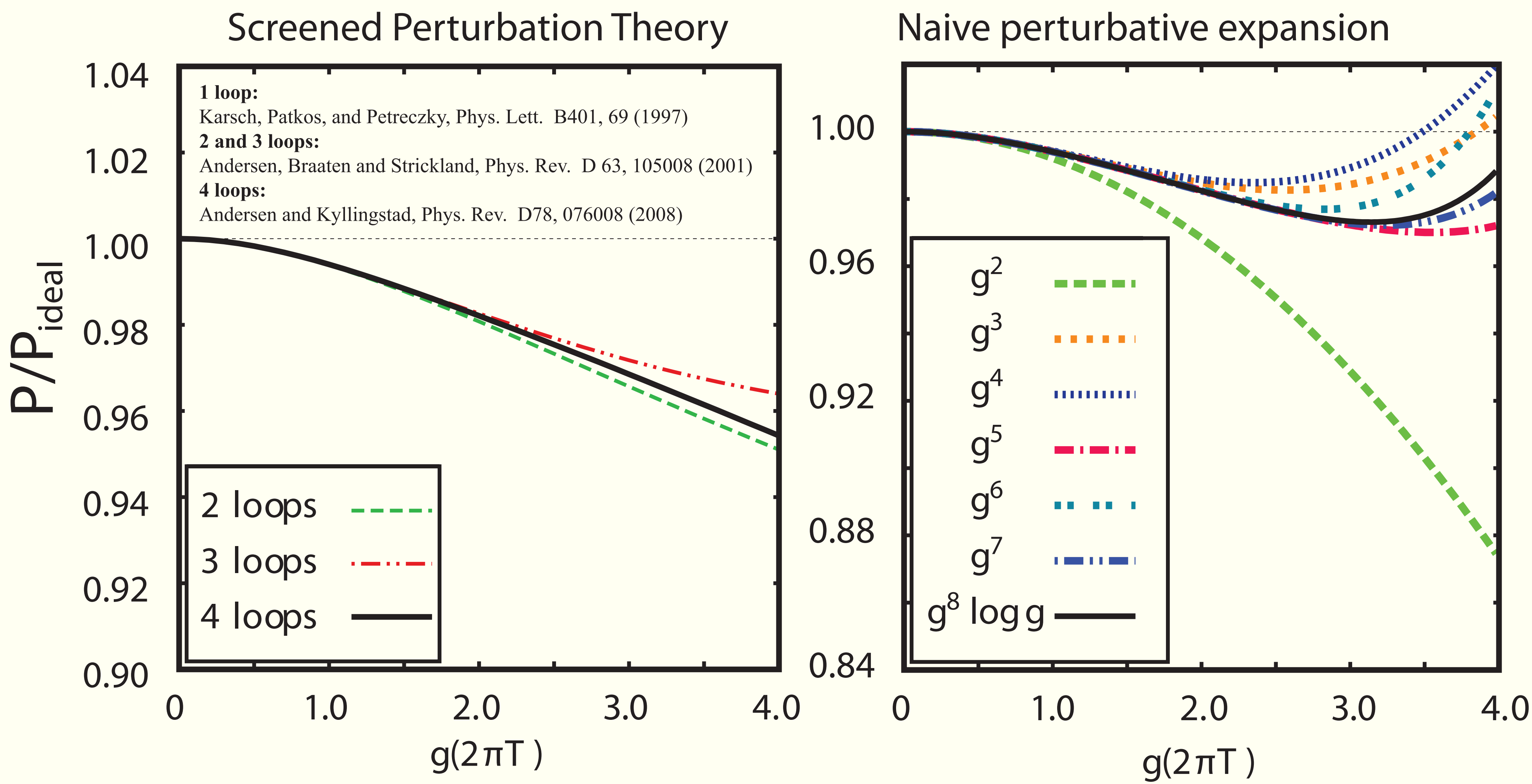
Trace Anomaly



Pressure – Higher Temperatures



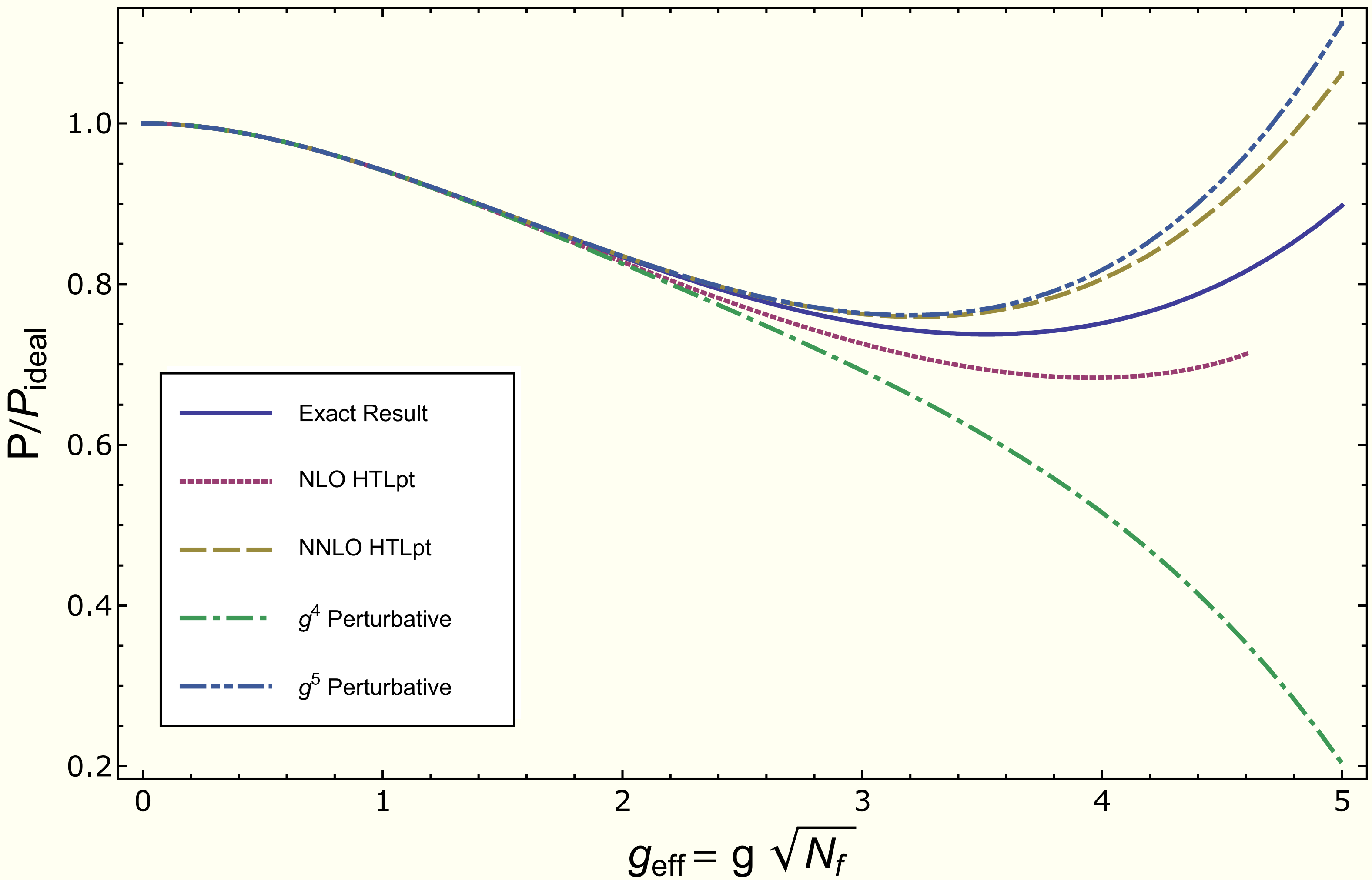
N^3LO Calculation in Scalar ϕ^4 Theory



Conclusions and Outlook

- The problem of bad convergence of finite temperature weak-coupling expansion is generic.
- It does not just occur in gauge theories, but also in scalar theories, and even in quantum mechanics.
- Variational perturbation theory and hard-thermal-loop perturbation theory can improve the convergence of perturbative calculations in a gauge-invariant manner which is formulated in Minkowski space.
- The NNLO results for pure-glue SU(3) Yang-Mills look very good for $T > 2 - 3\,T_c$! Especially considering that there are *no free parameters* to play with.
- Once the NNLO full QCD thermodynamics is obtained (COMING SOON!) we could start trying to use the HTLpt reorganization to calculate dynamic quantities such as momentum diffusion, viscosities, etc.

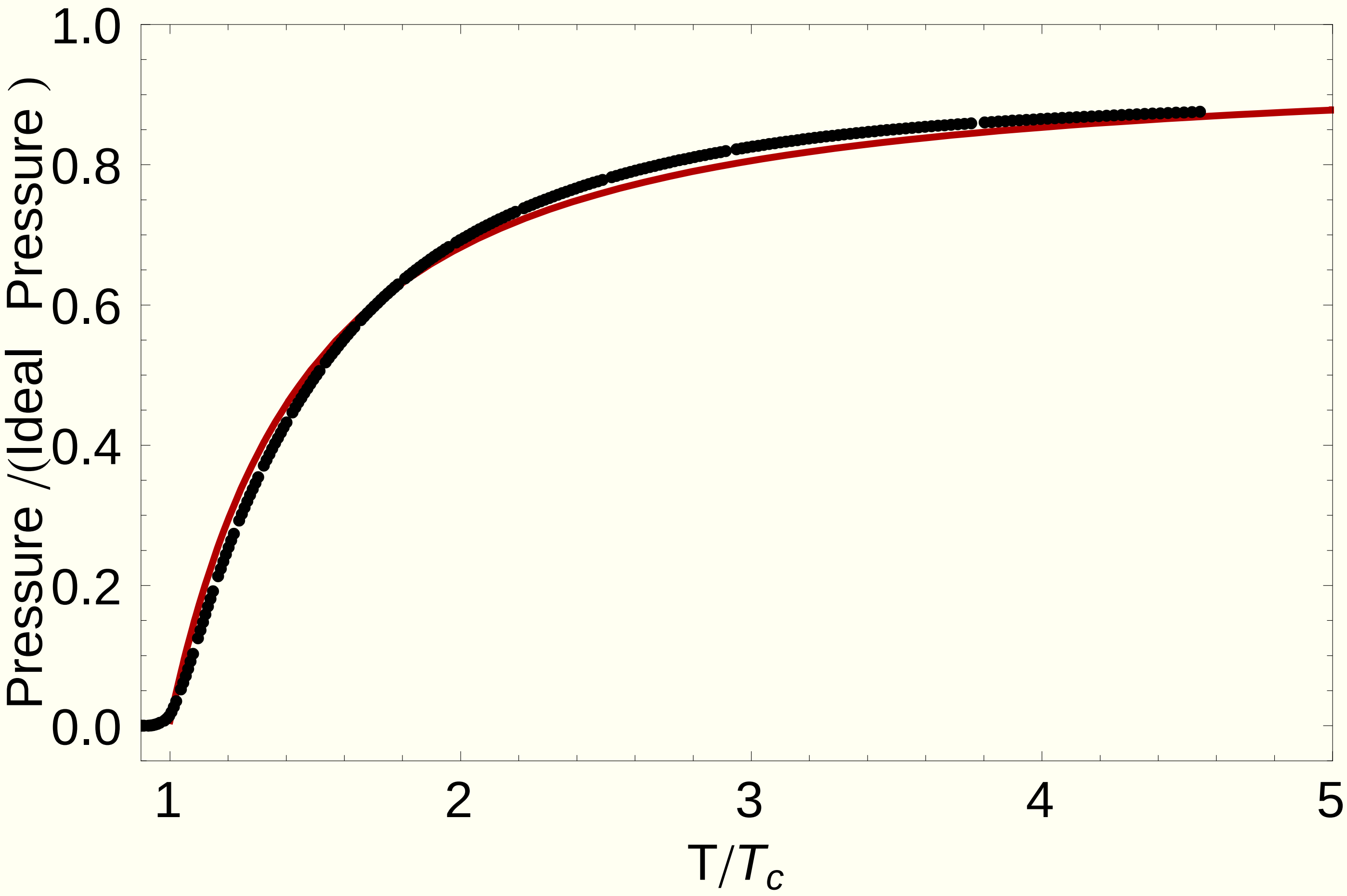
Large N_f Result - Comparison with Analytic Result



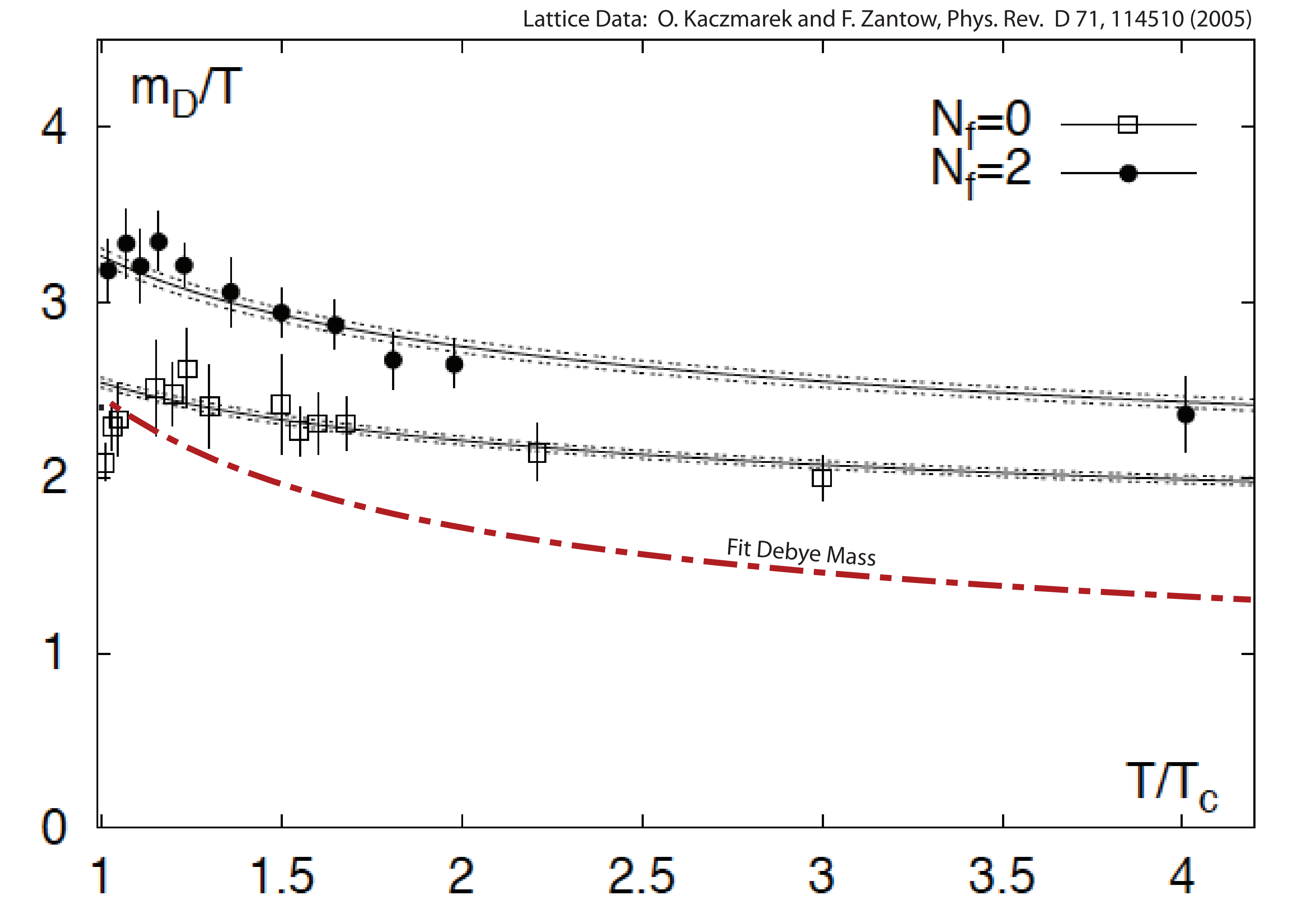
Exact result from Moore, Ipp, Rebhan.

Give me some fit parameters?

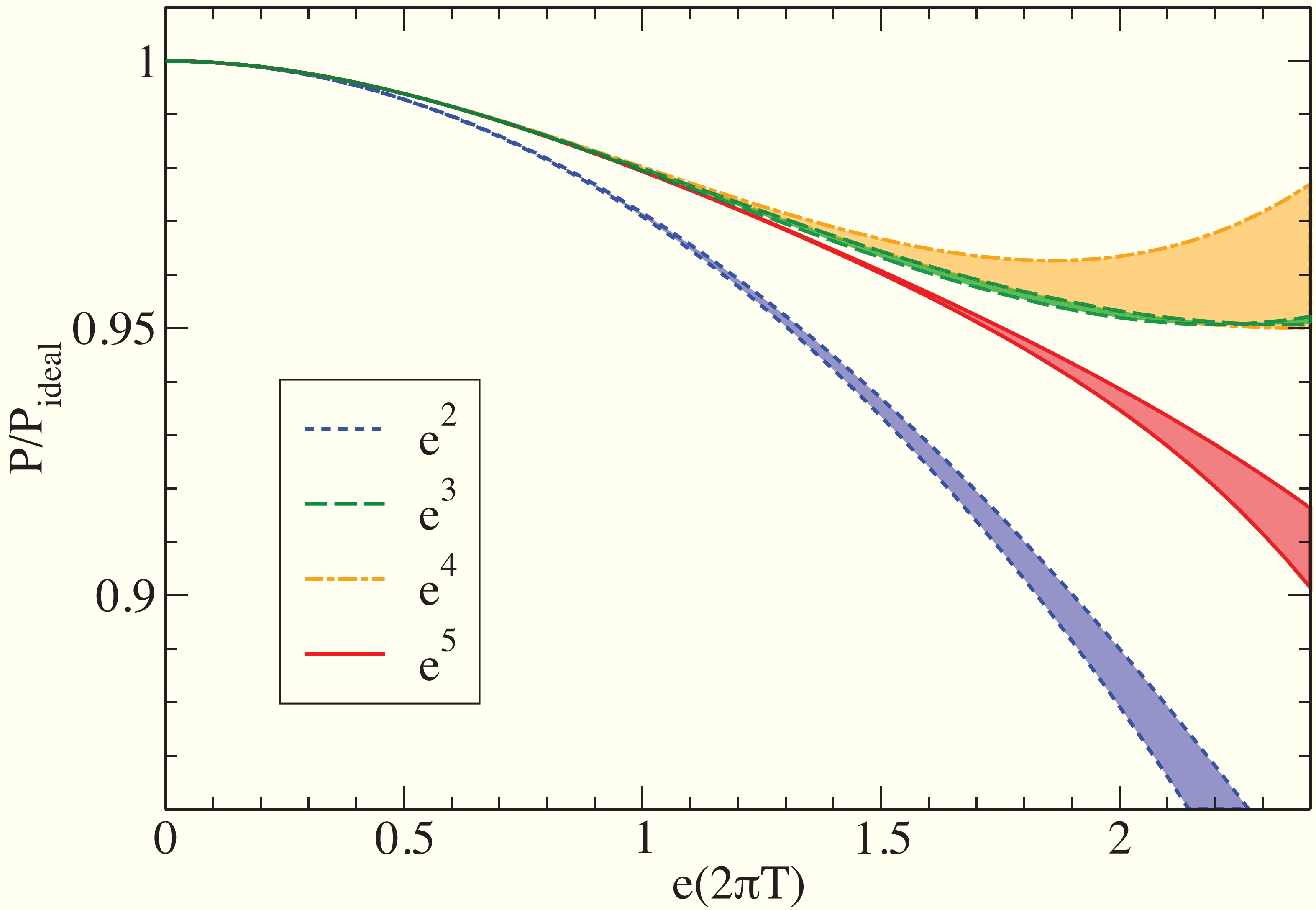
Using a “perturbative” ansatz for the Debye mass of the form $\hat{m}_D^2 = (\hat{m}_D^2)_{\rm LO}\left[1 + c_1 g^2 + c_2 g^4\right]$ I fit the pressure “by eye” with $c1 = -1.01$ and $c2 = 0.678$ to obtain the following plot. Could be improved if I spent more time fine tuning . . .



Is the resulting Debye mass consistent with lattice data?



HTLpt: naive pert. expansion of QED free energy



Perturbative QED free energy

HTLpt: 3-loop thermodynamic potential for QED

- The NNLO thermodynamic potential reads

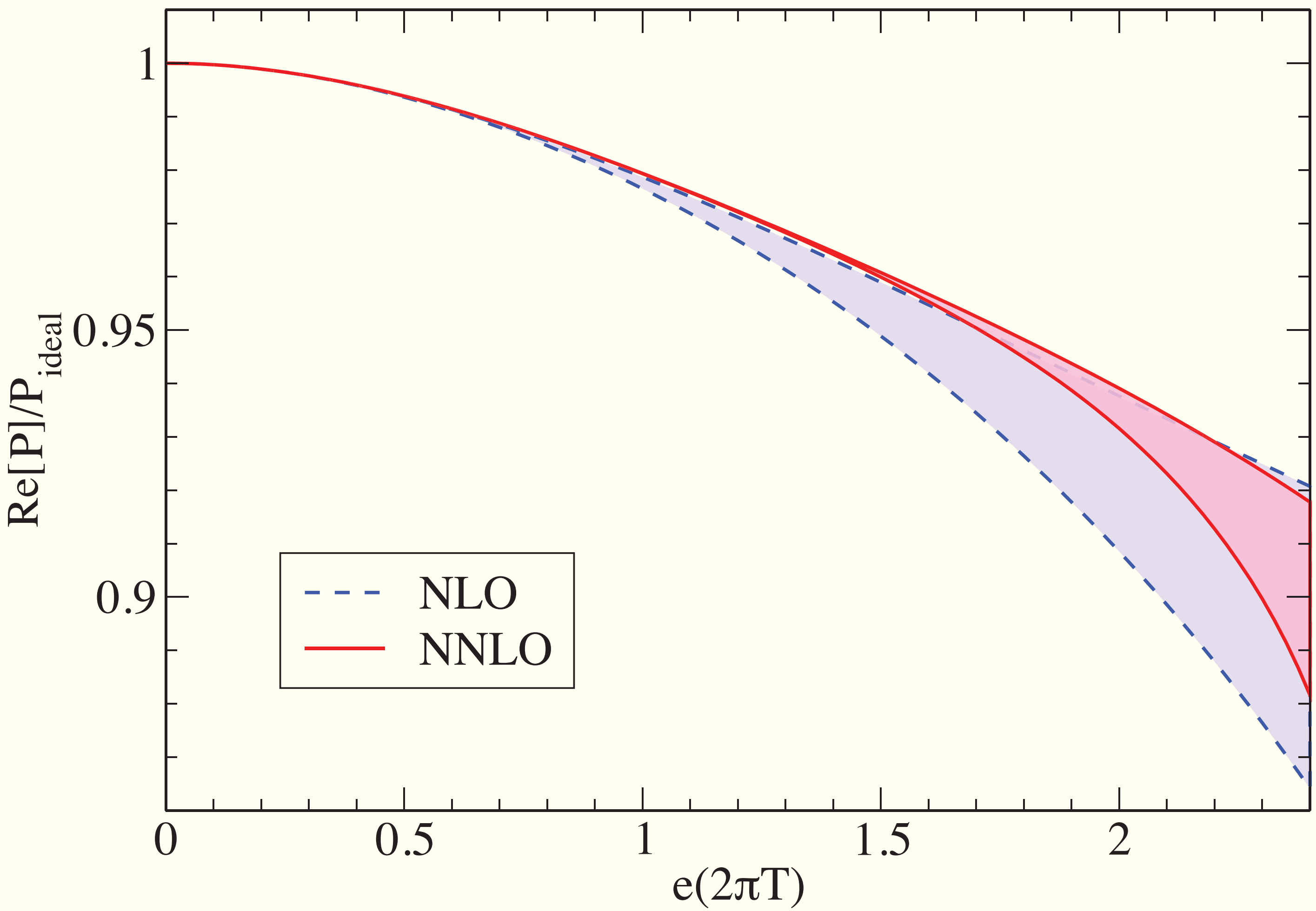
$$
\begin{aligned}
\Omega_{\rm NNLO} \quad = \quad & -\frac{\pi^2 T^4}{45}\Big\{1+\frac{7}{4}N_f-\frac{15}{4}\hat{m}_D^3 \\
& +N_f\frac{\alpha}{\pi}\Big[-\frac{25}{8}+\frac{15}{2}\hat{m}_D+15\Big(\log\frac{\hat{\mu}}{2}-\frac{1}{2}+\gamma+2\log 2\Big)\hat{m}_D^3-90\hat{m}_D\hat{m}_f^2\Big] \\
& +N_f\Big(\frac{\alpha}{\pi}\Big)^2\Big[\frac{15}{64}(35-32\log 2)-\frac{45}{2}\hat{m}_D\Big] \\
& +N_f^2\Big(\frac{\alpha}{\pi}\Big)^2\Big[\frac{25}{12}\Big(\log\frac{\hat{\mu}}{2}+\frac{1}{20}+\frac{3}{5}\gamma-\frac{66}{25}\log 2+\frac{4}{5}\frac{\zeta'(-1)}{\zeta(-1)}-\frac{2}{5}\frac{\zeta'(-3)}{\zeta(-3)}\Big) \\
& \qquad +\frac{5}{4}\frac{1}{\hat{m}_D}-15\Big(\log\frac{\hat{\mu}}{2}-\frac{1}{2}+\gamma+2\log 2\Big)\hat{m}_D\Big]\Big\}
\end{aligned}
$$

PURELY ANALYTIC!

- To eliminate the m_D and m_f dependence, the gap equations are imposed

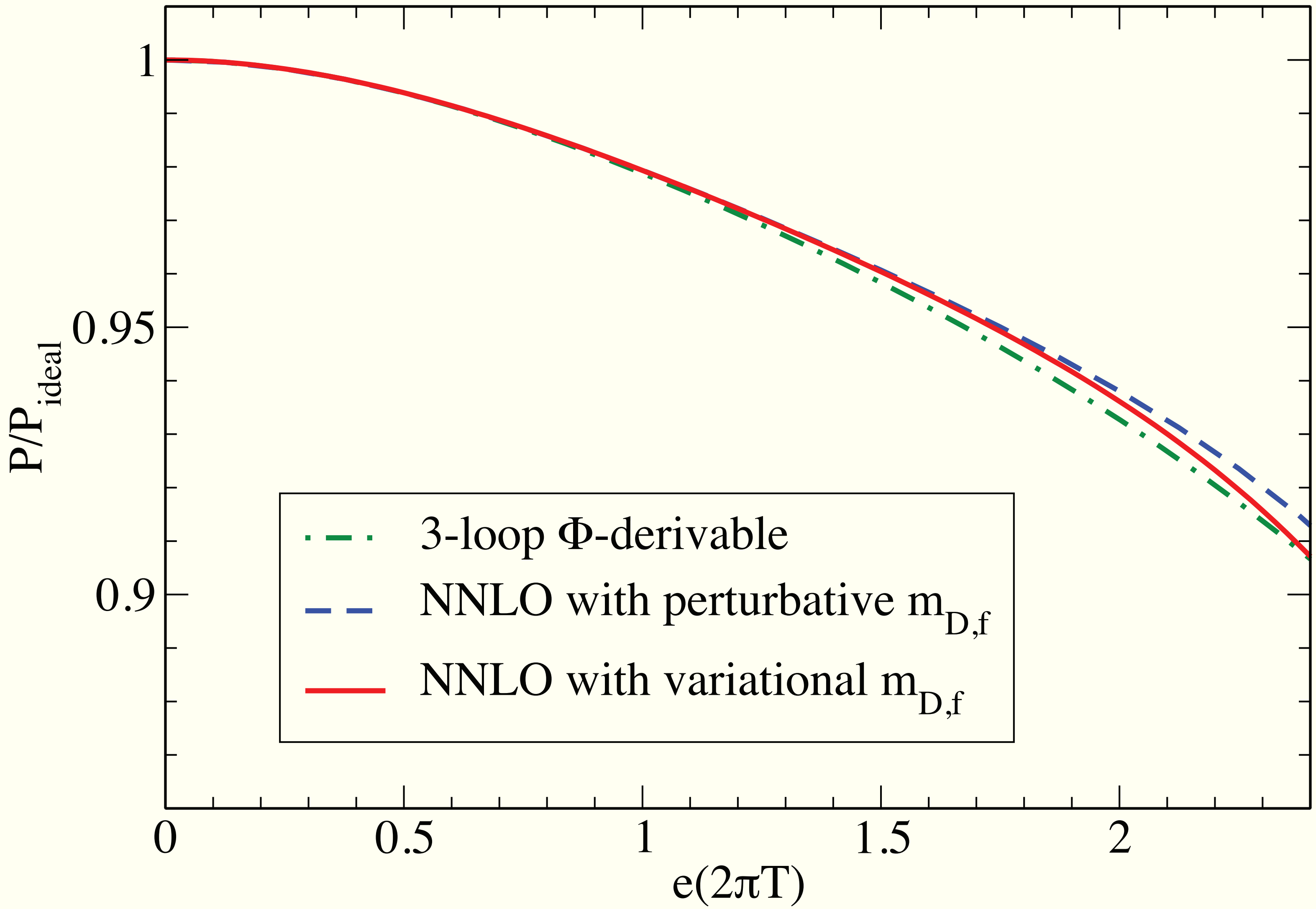
$$
\begin{aligned}
\frac{\partial}{\partial m_D}\Omega(T,\alpha,m_D,m_f,\delta=1) &= 0 \\
\frac{\partial}{\partial m_f}\Omega(T,\alpha,m_D,m_f,\delta=1) &= 0
\end{aligned}
$$

HTLpt: 2- and 3-loop free energy for QED



NLO and NNLO HTLpt predictions for QED free energy

HTLpt: comparison of different methods/schemes



Comparison of three different predictions for the QED free energy at $\mu = 2\pi T$

3-loop Φ-derivable result is taken from Andersen and Strickland, 05