
Hard loop effective theory and quark-gluon plasma instabilities

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work in collaboration with:

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Quark-gluon plasma at RHIC

Surprises at RHIC (judging from hydro simulations):

- Very early thermalization/isotropization
- Very low shear viscosity

New paradigm: **sQGP**

(very successful toy model:

maximally supersymmetric large- N_c YM theory at infinite 't Hooft coupling from AdS/CFT)

But:

wQGP (thermal pQCD)

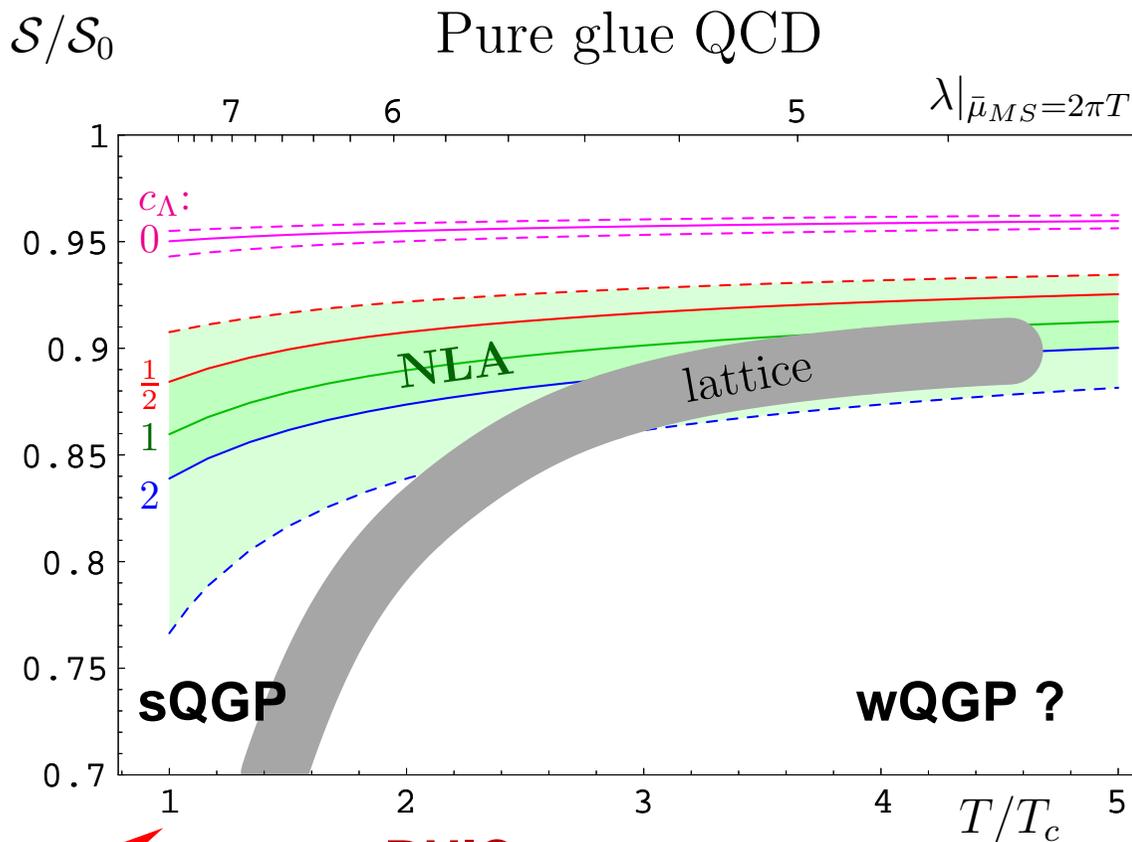
not yet fully understood, especially far from equilibrium!

furthermore: RHIC close to phase transition (reason for sQGP?)

LHC will reach $\gtrsim 3T_c$ — wQGP eventually there?

wQGP or sQGP?

Entropy in pure-gluon QCD: lattice vs. **H**ard-**T**hermal-**L**oop quasiparticle entropy with **N**ext-to-**L**eading **A**pproximations of asymptotic thermal masses suggestive of dominance of weakly interacting (hard) quasiparticles for $T \gtrsim 3T_c$



[Blaizot, Iancu & AR, PRD63('01)065003]

$$\alpha_s(\bar{\mu} = \pi T \dots 4\pi T)$$

from *standard* 2-loop pQCD !

similar picture with
perturbative result to order g^5
(Kajantie, Laine, Rummukainen, Schröder)
and NNLO HTLpt
(Andersen, Braaten, Strickland, Su)

cp. talk by

Mike Strickland

← RHIC

← LHC

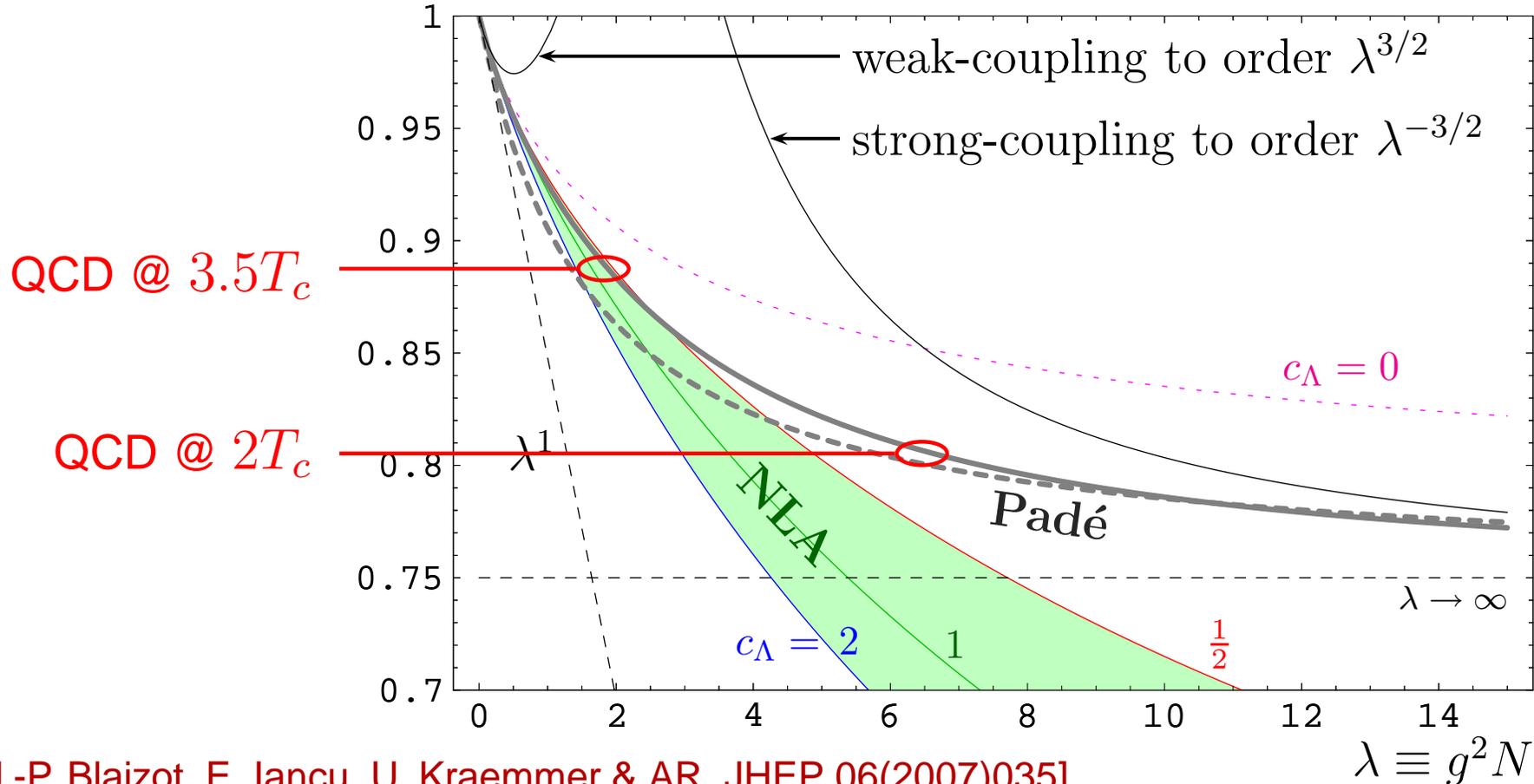
wQGP or sQGP?

AdS/CFT: No lattice EOS results for $\mathcal{N} = 4$ SYM,

but essentially unique Padé approximant $R_{[4,4]} = \frac{1 + \alpha\lambda^{1/2} + \beta\lambda + \gamma\lambda^{3/2} + \delta\lambda^2}{1 + \bar{\alpha}\lambda^{1/2} + \bar{\beta}\lambda + \bar{\gamma}\lambda^{3/2} + \bar{\delta}\lambda^2}$

for known weak and strong coupling results

S/S_0 $\mathcal{N} = 4$ super-Yang-Mills



[J.-P. Blaizot, E. Iancu, U. Kraemmer & AR, JHEP 06(2007)035]

wQGP or sQGP?

- Truth probably in between wQGP and sQGP
- Need to understand wQGP in systematic limit $g \ll 1$
(*really* weakly coupled QGP)
 - theoretical challenge (sQGP in many ways simpler)
 - use for bold extrapolation to $g \sim 1$

Scales of wQGP

- T : energy of hard particles
- gT : thermal masses, Debye screening mass, Landau damping
- g^2T : magnetic confinement, color relaxation, rate for small angle scattering
- g^4T : rate for large angle scattering; inverse shear viscosity $\eta^{-1}T^4$

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Effective theory at scale gT : **Hard-Thermal-Loop Effective Action**

[Frenkel, Taylor & Wong; Braaten & Pisarski 1991]

equivalent to: **gauge-covariant Boltzmann-Vlasov**

[Blaizot & Iancu 1993, Kelly, Liu, Lucchesi & Manuel 1994]

in particular required (to leading order!) for:

- Bottom-up thermalization [Baier, Mueller, Schiff & Son 2000]

$$t_{eq} \propto g^{-13/5}$$

- Shear viscosity [Arnold, Moore & Yaffe]

$$(\eta/s)^{-1} = g^4 \ln(1/g) f(\ln(1/g))$$

Scales of wQGP

- T : energy of hard particles
- gT : thermal masses, Debye screening mass, Landau damping, **plasma instabilities** [Mrówczyński 1988, 1993, ...]
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$$t_{eq} \propto g^{-13/5} \rightarrow g^{-?} \quad [\text{Arnold, Lenaghan, Moore, JHEP 08 ('03) 002}]$$

- Shear viscosity [Arnold, Moore & Yaffe]

$$(\eta/s)^{-1} = g^4 \ln(1/g) f(\ln(1/g)) \quad \text{add } (\eta/s)_{\text{anomalous}}^{-1} !$$

[Asakawa, Bass & Müller, PRL 96 ('06) 252301]

Hard (Thermal) Loops = gauge covariant Boltzmann-Vlasov

With color-neutral background distribution $v \cdot \partial f_0(\mathbf{p}, \mathbf{x}, t) = 0$, $v^\mu = p^\mu / p^0$

$$v \cdot D \delta f_a(\mathbf{p}, \mathbf{x}, t) = g v_\mu F_a^{\mu\nu} \partial_\nu^{(p)} f_0(\mathbf{p}, \mathbf{x}, t) = -g(\mathbf{E}_a + \mathbf{v} \times \mathbf{B}_a) \cdot \nabla_{\mathbf{p}} f_0,$$

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- anisotropic expansion: $f_0(\mathbf{p}; \mathbf{x}, t)$

Hard loop gauge boson self energy

HTL effective theory nonlinear!

$$j^\mu = j^\mu[A] = \int \Pi^{\mu\nu} A_\nu + \frac{1}{2} \int \int \Gamma^{\mu\nu\lambda} A_\nu A_\lambda + \dots \text{ ad infinitum}$$

Linearize in A^μ and Fourier transform

$$j^\mu(k) = g^2 \int \frac{d^3 p}{(2\pi)^3} v^\mu \underbrace{\partial_{(p)}^\beta f(\mathbf{p})}_{0 \text{ for } \beta=0} \left(g_{\gamma\beta} - \frac{v_\gamma k_\beta}{k \cdot v + i\epsilon} \right) A^\gamma(k) = \Pi^{\mu\nu}(k) A_\nu(k)$$

$i\epsilon \leftrightarrow$ retarded boundary condition

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Isotropic case: $\partial_{(p)}^\beta f(|\mathbf{p}|) = f'(|\mathbf{p}|) (0, p^b / |\mathbf{p}|)$

\rightarrow 2 functions $\Pi_T(k_0/|\mathbf{k}|)$, $\Pi_L(k_0/|\mathbf{k}|) \propto m^2 = g^2 p_{\text{hard}}^2$

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Generic case:

10 - 4 = 6 functions, each depending on 3 variables k_i/k_0

Isotropic gauge boson self energy

Gauge invariance of HTL/HDL effective action \rightarrow transverse gauge boson self energy

2 tensors transverse w.r.t. 4-momentum in a thermal medium (rest frame velocity $u^\mu = \delta_0^\mu$)

$$A_{\mu\nu} = g_{\mu\nu} - \frac{k_\mu k_\nu}{k^2} - B_{\mu\nu},$$

$$B_{\mu\nu} = \frac{\tilde{n}_\mu \tilde{n}_\nu}{\tilde{n}^2} \text{ with } \tilde{n}_\mu = \left(g_{\mu\nu} - \frac{k_\mu k_\nu}{k^2}\right) u^\nu$$

$$\Pi_A \equiv \Pi_T = \frac{1}{2} A_{\mu\nu} \Pi^{\mu\nu} = \frac{1}{2} (\Pi^\mu{}_\mu - \Pi_B)$$

$$\Pi_B \equiv \Pi_L = -\frac{k^2}{\mathbf{k}^2} \Pi_{00}$$

$$\Pi^\mu{}_\mu = m_D^2, \quad \Pi_{00} = m_D^2 \left(1 - \frac{k^0}{2|\mathbf{k}|} \ln \frac{k^0 + |\mathbf{k}|}{k^0 - |\mathbf{k}|}\right)$$

Gauge boson propagator (Landau gauge)

$$-G_{\mu\nu} = \Delta_T A_{\mu\nu} + \Delta_L B_{\mu\nu}$$

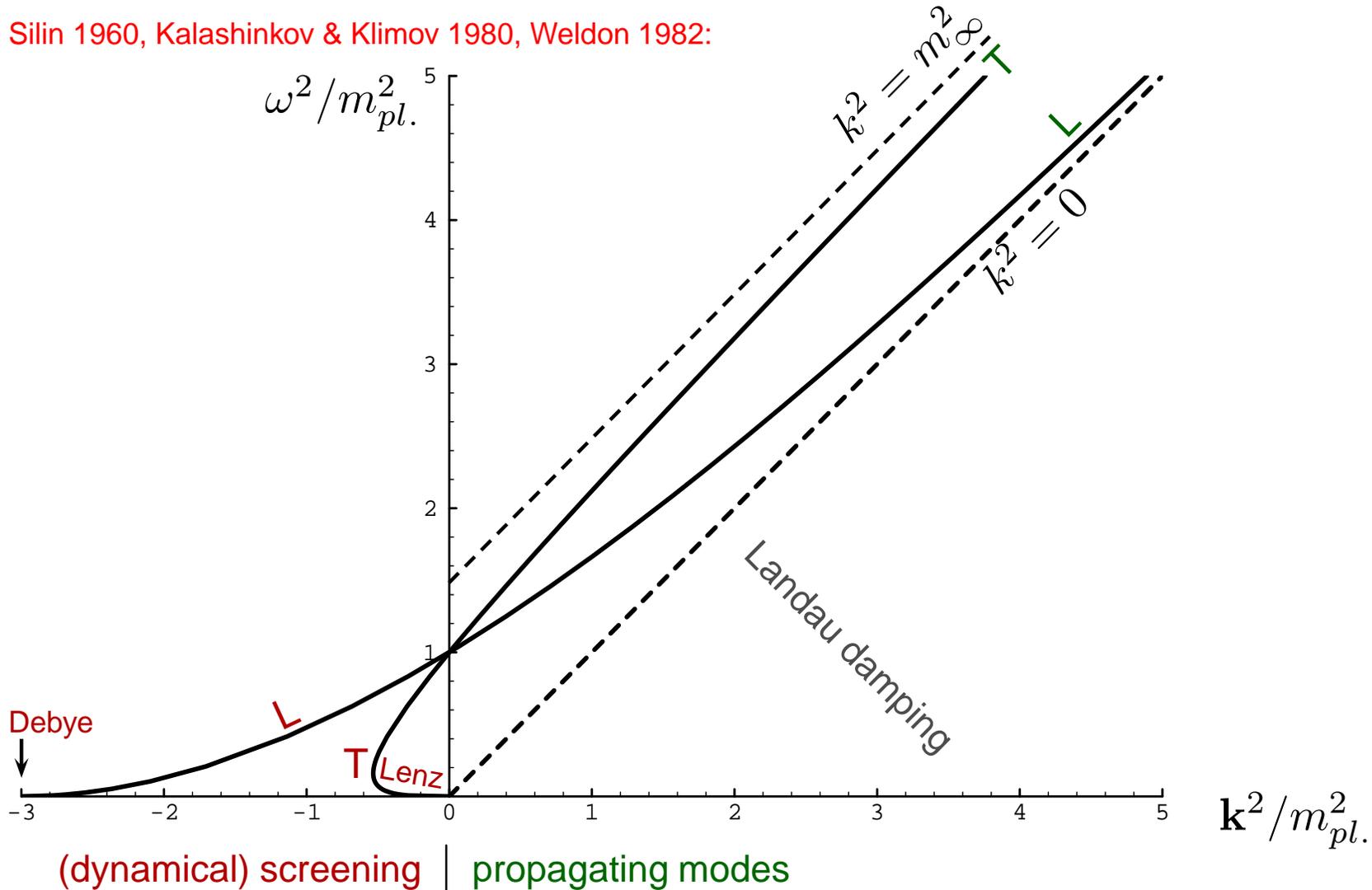
$$\Delta_T = [k^2 - \Pi_T]^{-1}, \quad \Delta_L = [k^2 - \Pi_L]^{-1}$$

\rightarrow 2 branches with different dispersion laws

Dispersion laws of HTL/HDL gauge bosons

Isotropic case (not necessarily thermal)

Silin 1960, Kalashinkov & Klimov 1980, Weldon 1982:



Hard anisotropic loop gauge boson self energy

$$\Pi^{\mu\nu}(k) = g^2 \int \frac{d^3p}{(2\pi)^3} v^\mu \partial_\beta^{(p)} f(\mathbf{p}) \left(g^{\nu\beta} - \frac{v^\nu k^\beta}{k \cdot v + i\epsilon} \right), \quad v^\mu \equiv \frac{p^\mu}{p^0}, \quad p^0 = |\mathbf{p}|$$

$\Pi^{\mu\nu}$ symmetric, $\Pi^{0\nu}$ fixed by transversality $k_\mu \Pi^{\mu\nu} = 0 \rightarrow 6$ structure functions in general

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Assume just one direction of anisotropy (axisymmetry): $\mathbf{n} = (0, 0, 1)$

$\rightarrow 4$ symmetric tensors for Π^{ij} , 4 independent structure functions

$$A^{ij} = \delta^{ij} - k^i k^j / k^2, \quad B^{ij} = k^i k^j / k^2,$$

$$C^{ij} = \tilde{n}^i \tilde{n}^j / \tilde{n}^2, \quad D^{ij} = k^i \tilde{n}^j + k^j \tilde{n}^i, \quad \tilde{n}^i = A^{ij} n^j$$

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Propagator (temporal axial gauge $A^0 = 0$ for simplicity)

$$\Delta(K) = \Delta_T \mathbf{A} + (k^2 - \omega^2 + \alpha + \gamma) \Delta_{\mathcal{L}} \mathbf{B} + [(\beta - \omega^2) \Delta_{\mathcal{L}} - \Delta_T] \mathbf{C} - \delta \Delta_{\mathcal{L}} \mathbf{D}$$

$$\Delta_T(k) = [k^2 - \omega^2 + \alpha]^{-1}$$

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$$\Delta_{\mathcal{L}}(k) = [\cancel{(k^2 - \omega^2 + \alpha + \gamma)} (\beta - \omega^2) - \cancel{k^2 \tilde{n}^2 \delta^2}]^{-1}$$

generally: 2 branches from $\Delta_{\mathcal{L}}$; only 1 from $\Delta_{\mathcal{L}}$ when $\mathbf{k} \parallel \mathbf{n} \Rightarrow \tilde{n} = 0$

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Special important case: $f(\mathbf{p}) = f_{\text{iso}} (\mathbf{p}^2 + \xi(\mathbf{p} \cdot \mathbf{n})^2)$

$\xi = 0$: isotropic; $-1 < \xi < 0$: prolate (cigar-shaped); $0 < \xi < \infty$: oblate (squashed)

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Can be evaluated in closed form: **[Romatschke & Strickland 2003]**

Change variables $\mathbf{p}^2 + \xi(\mathbf{p} \cdot \mathbf{n})^2 = \bar{p}^2$

$$\Pi^{ij}(k) = m^2 \int \frac{d\Omega}{4\pi} v^i \frac{v^l + \xi(\mathbf{v} \cdot \mathbf{n})n^l}{(1 + \xi(\mathbf{v} \cdot \mathbf{n})^2)^2} \left(\delta^{jl} + \frac{v^j k^l}{k \cdot v + i\epsilon} \right)$$

$$m^2 \equiv -\frac{g^2}{2\pi^2} \int_0^\infty d\bar{p} \bar{p}^2 \frac{df_{\text{iso}}(\bar{p}^2)}{d\bar{p}}$$

Magnetostatic polarization function for $\mathbf{k} \parallel \mathbf{n}$

Static limit: $\alpha(k) \equiv \Pi_T \rightarrow \frac{1}{2}\Pi^{ii}(\omega = 0, \mathbf{k} \cdot \mathbf{n}/k)$ because then $k^i \Pi^{ij} \rightarrow 0$

Easy exercise: calculate $\Pi^{ii}(\omega = 0)$ for $\mathbf{k} \parallel \mathbf{n}$!

Solution:

$$\begin{aligned}\alpha/m^2 &= \frac{1}{4} \left[(1 - \xi) \frac{\arctan \sqrt{\xi}}{\sqrt{\xi}} - 1 \right] && \text{for } \xi > 0 \\ &= \frac{1}{4} \left[(1 - \xi) \frac{\operatorname{atanh} \sqrt{-\xi}}{\sqrt{-\xi}} - 1 \right] && \text{for } \xi < 0\end{aligned}$$

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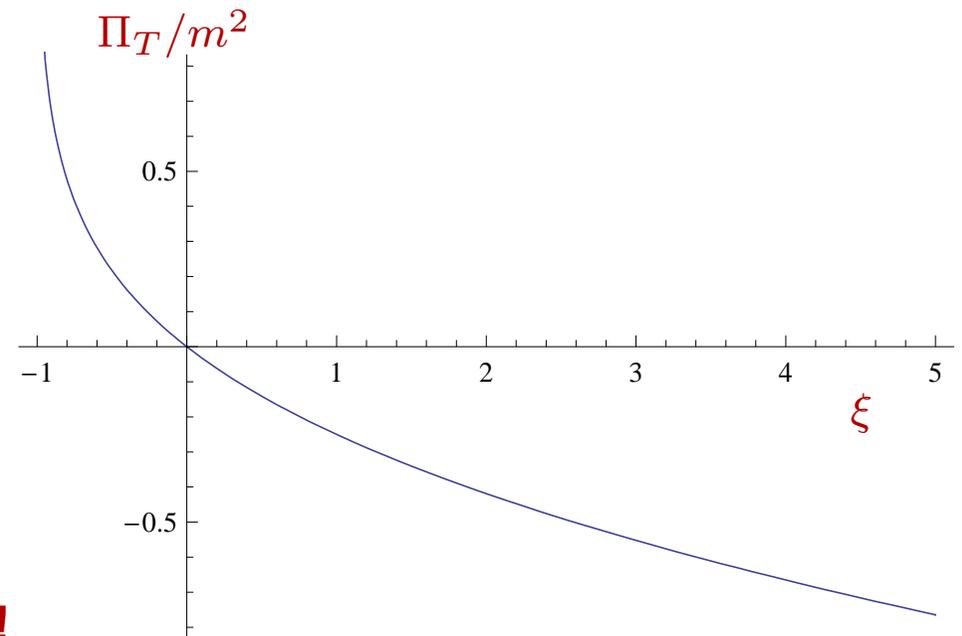
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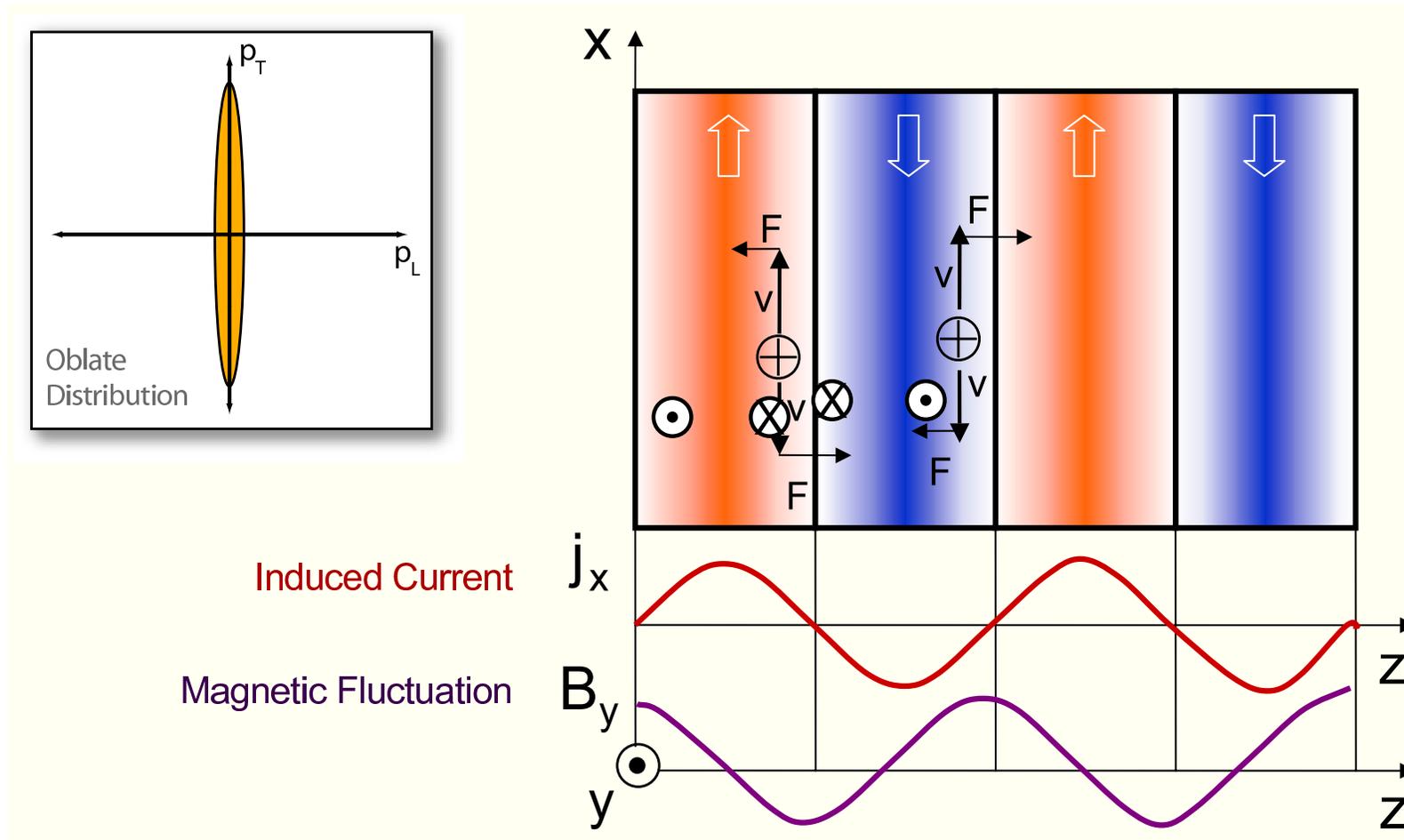
$\alpha = \Pi_T$ is magnetic screening mass

- $\xi = 0$ (isotropic):
no magnetic screening mass
- $\xi < 0$ (prolate):
magnetostatic screening!
- $\xi > 0$ (oblate):
“tachyonic” magnetic mass — **instability!**



Filamentation (Weibel) instabilities

E.g.: ensemble of counterstreaming currents unstable against filamentation



Abelian plasma: exponential growth of currents and magnetic fields until magnetic fields strong enough to bend trajectories \rightarrow fast isotropization

Full anisotropic polarization tensor for $\mathbf{k} \parallel \mathbf{n}$

For full dispersion laws (for $\mathbf{k} \parallel \mathbf{n}$ which contains the most unstable modes) need complete frequency dependences ($\eta \equiv \omega/k$) [Romatschke & Strickland 2004]

$$\alpha = \frac{m^2}{4\sqrt{\xi}(1+\xi\eta^2)^2} \left[(1 + \eta^2 + \xi(-1 + (6 + \xi)\eta^2 - (1 - \xi)\eta^4)) \arctan \sqrt{\xi} \right. \\ \left. + \sqrt{\xi}(\eta^2 - 1) \left(1 + \xi\eta^2 - (1 + \xi)\eta \ln \frac{\eta + 1 + i\epsilon}{\eta - 1 + i\epsilon} \right) \right],$$

$$\beta = -\frac{\eta^2 m^2}{2\sqrt{\xi}(1+\xi\eta^2)^2} \left[(1 + \xi)(1 - \xi\eta^2) \arctan \sqrt{\xi} \right. \\ \left. + \sqrt{\xi} \left((1 + \xi\eta^2) - (1 + \xi)\eta \ln \frac{\eta + 1 + i\epsilon}{\eta - 1 + i\epsilon} \right) \right]$$

more complicated: $\mathbf{k} \not\parallel \mathbf{n}$

- second branch of poles in $\Delta_{\mathcal{L}}$ which can contain *electric* (Buneman) instability

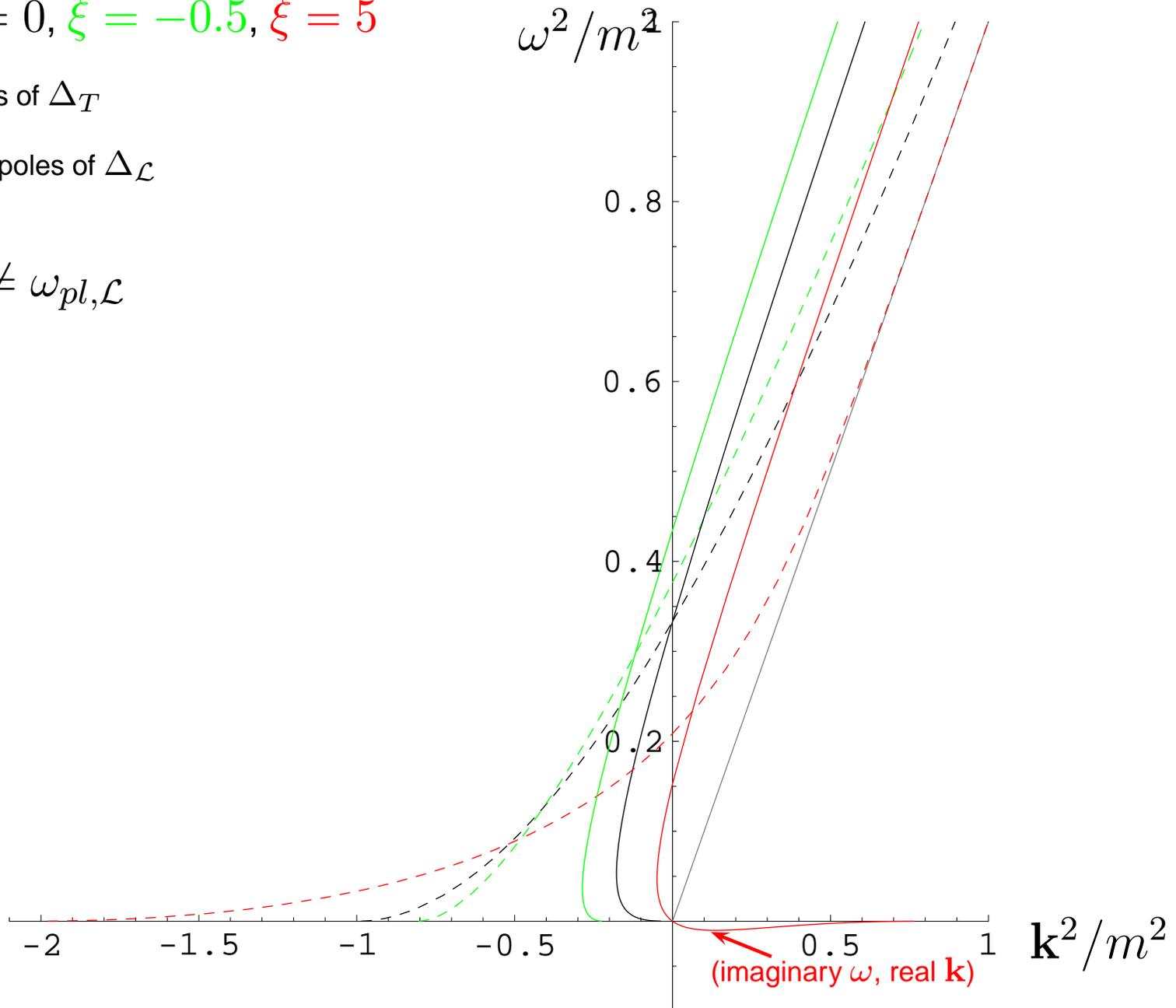
Dispersion laws for $\mathbf{k} \parallel \mathbf{n}$

Comparing $\xi = 0$, $\xi = -0.5$, $\xi = 5$

full lines: poles of Δ_T

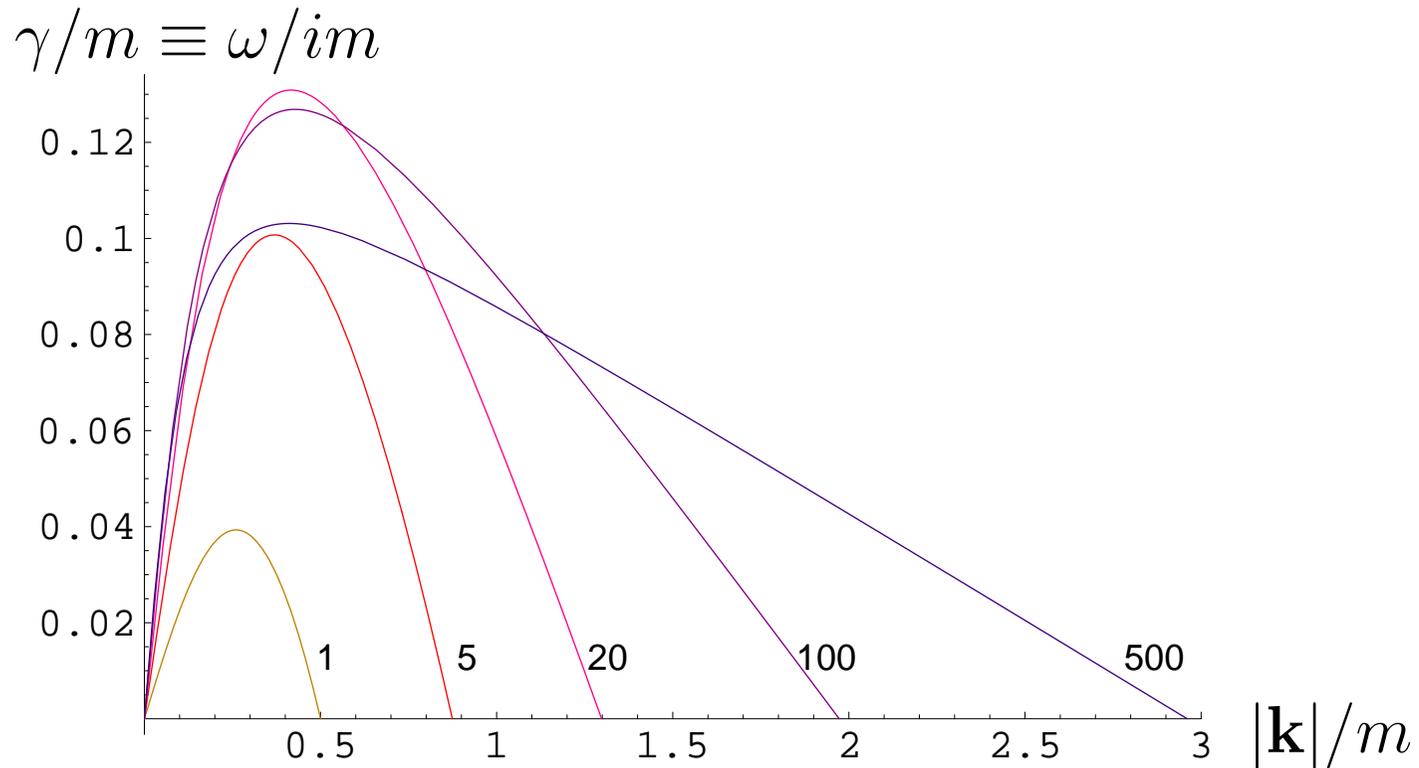
dashed lines: poles of $\Delta_{\mathcal{L}}$

$\xi \neq 0$: $\omega_{pl,T} \neq \omega_{pl,\mathcal{L}}$



Growth rates for Weibel instabilities ($\mathbf{k} \parallel \mathbf{n}$)

Anisotropy parameter $\xi = 1, 5, 20, 100, 500$ (increasing oblateness)



large ξ behavior: $k_{\max}/m \sim \xi^{1/4}$, $k/m|_{\gamma=\gamma_{\max}} \sim 1$

compared to asymptotic gluon mass m_{∞} : $k_{\max}/m_{\infty} \sim \sqrt{\xi}$

$$\gamma_{\max}/m_{\infty} \rightarrow 1/\sqrt{2}$$

Hard Anisotropic Loop Effective Action

Exponentially growing gluon fields

⇒ linear response regime quickly left

⇒ need the (infinitely many) vertex functions required by gauge invariance

Hard (Thermal) Loop Effective Action formally still given by

[Pisarski 1993, Mrówczyński, Strickland, AR 2004]

$$S_{\text{aniso}} = -\frac{g^2}{2} \int_x \int_{\mathbf{p}} \left\{ f(\mathbf{p}) F_{\mu\nu}^a(x) \left(\frac{p^\nu p^\rho}{(p \cdot D)^2} \right)_{ab} F_\rho^{b\mu}(x) \right\}$$

nonlinear and nonlocal!

— only useful as generating functional of hard anisotropic loops

Hard Loop Effective Theory

Useful:

auxiliary field formulation: [Nair; Blaizot & Iancu 1994; Mrówczyński, AR & Strickland 2004]

$$\delta f^a(x; p) = -g W_\mu^a(t, \mathbf{x}; \mathbf{v}) \partial_{(p)}^\mu f_0(\mathbf{p})$$

$$\boxed{[v \cdot D(A)] W_\mu(x; \mathbf{v}) = F_{\mu\gamma}(A) v^\gamma}$$

$$v^\mu \equiv p^\mu / |\mathbf{p}| = (1, \mathbf{v})$$

$$\boxed{D_\rho(A) F^{\rho\mu} = j^\mu(x) = -g^2 \int \frac{d^3 p}{(2\pi)^3} \frac{1}{2|\mathbf{p}|} p^\mu \frac{\partial f(\mathbf{p})}{\partial p^\nu} W^\nu(x; \mathbf{v})}$$

Hard Loop effective theory: (hard) scale $|\mathbf{p}|$ can be integrated out

Auxiliary field version: local in terms of field living also on velocity space S_2

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Hard Loop effective theory: (hard) scale $|\mathbf{p}|$ can be integrated out

Auxiliary field version: local in terms of field living also on velocity space S_2

Nonlinear response \rightarrow real-time lattice simulation

Discretized Hard Loop Effective Theory

Useful:

auxiliary field formulation: [Nair; Blaizot & Iancu 1994; Mrówczyński, AR & Strickland 2004]

$$\delta f^a(x; p) = -g W_\mu^a(t, \mathbf{x}; \mathbf{v}) \partial_{(p)}^\mu f_0(\mathbf{p})$$

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Hard Loop effective theory: (hard) scale $|\mathbf{p}|$ can be integrated out

Auxiliary field version: local in terms of field living also on velocity space S_2

Nonlinear response \rightarrow real-time lattice simulation

\rightarrow discretize also velocity space

$$D_\rho(A) F^{\rho\mu} = j^\mu(x) = \frac{1}{\mathcal{N}} \sum_{\mathbf{v}} v^\mu \mathcal{W}_{\mathbf{v}}(x)$$

“disco balls”



Transversely constant modes: 1D+3V

Most unstable modes in linear response: $\mathbf{k} \parallel \mathbf{n}$

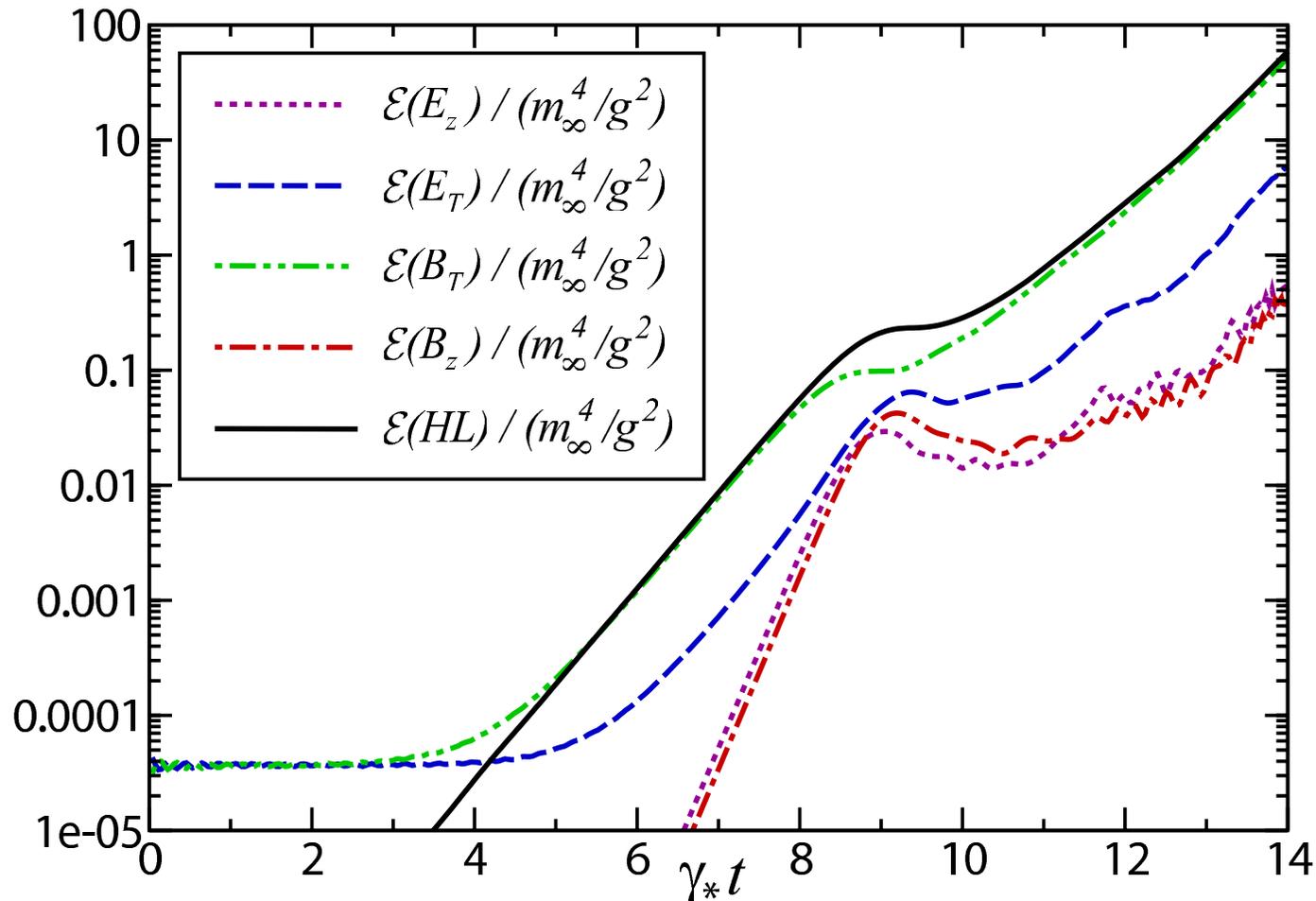
\implies no dependence on transverse coordinates;
dimensional reduction to 1 spatial dimension

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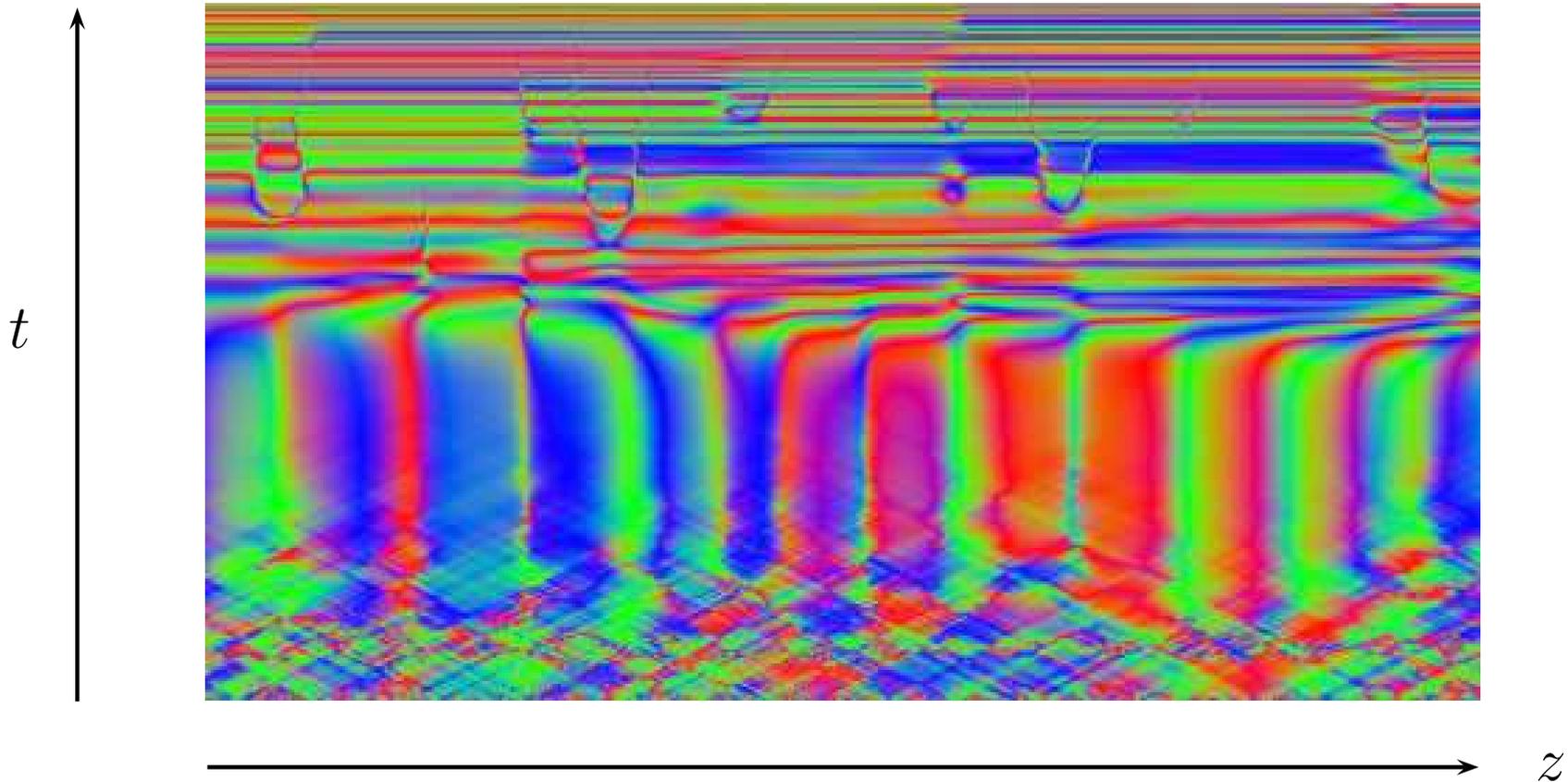
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[AR, Romatschke & Strickland, PRL 94 ('05) 102303]



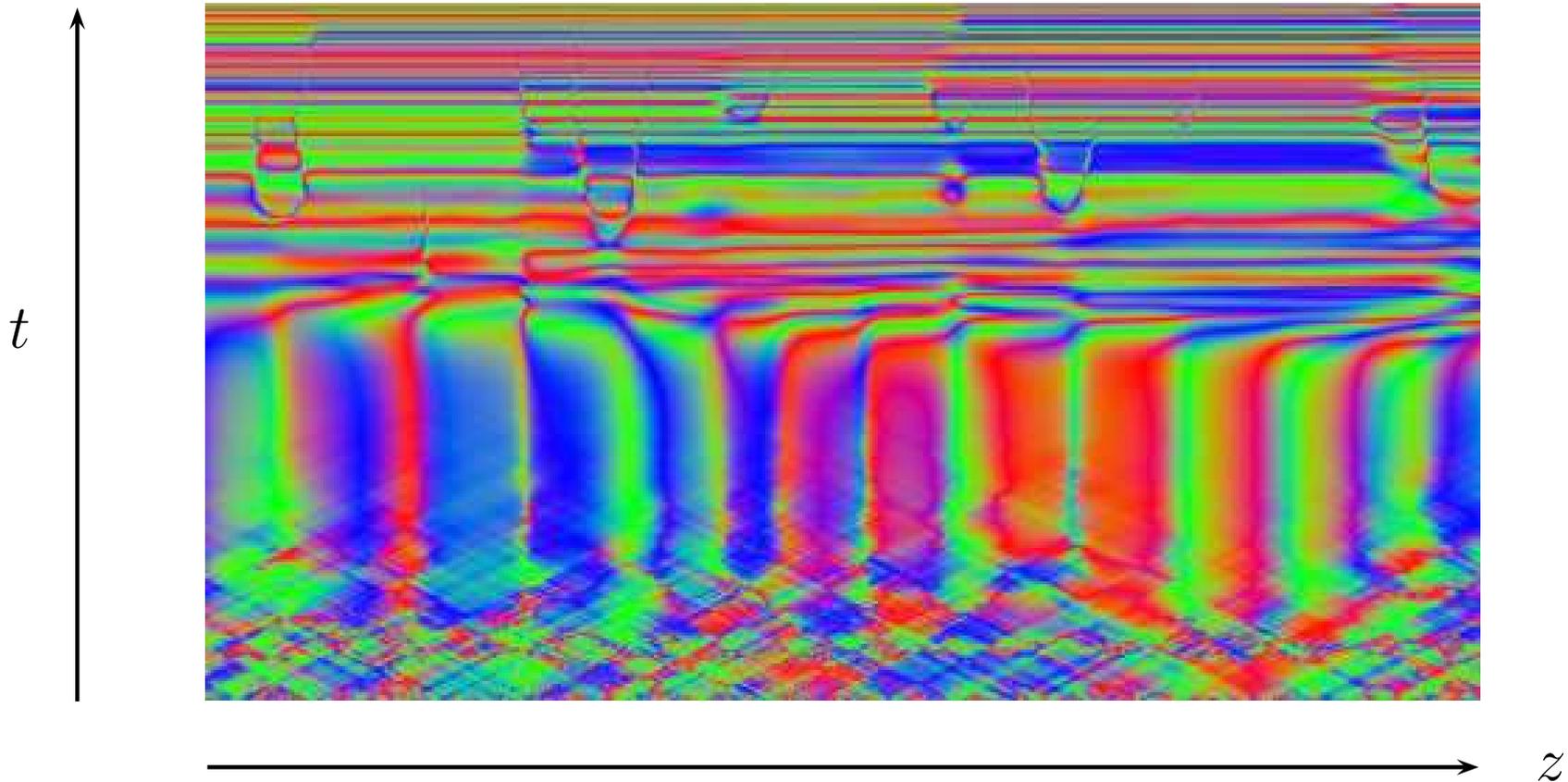
Transversely constant modes: 1D+3V

Evolution of color degrees of freedom:
(parallel-transported color from fixed spatial point)



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Late-time (non-linear) regime: Abelianization over extended spatial domains
– responsible for continued Abelian-like growth in non-linear regime

3D+3V

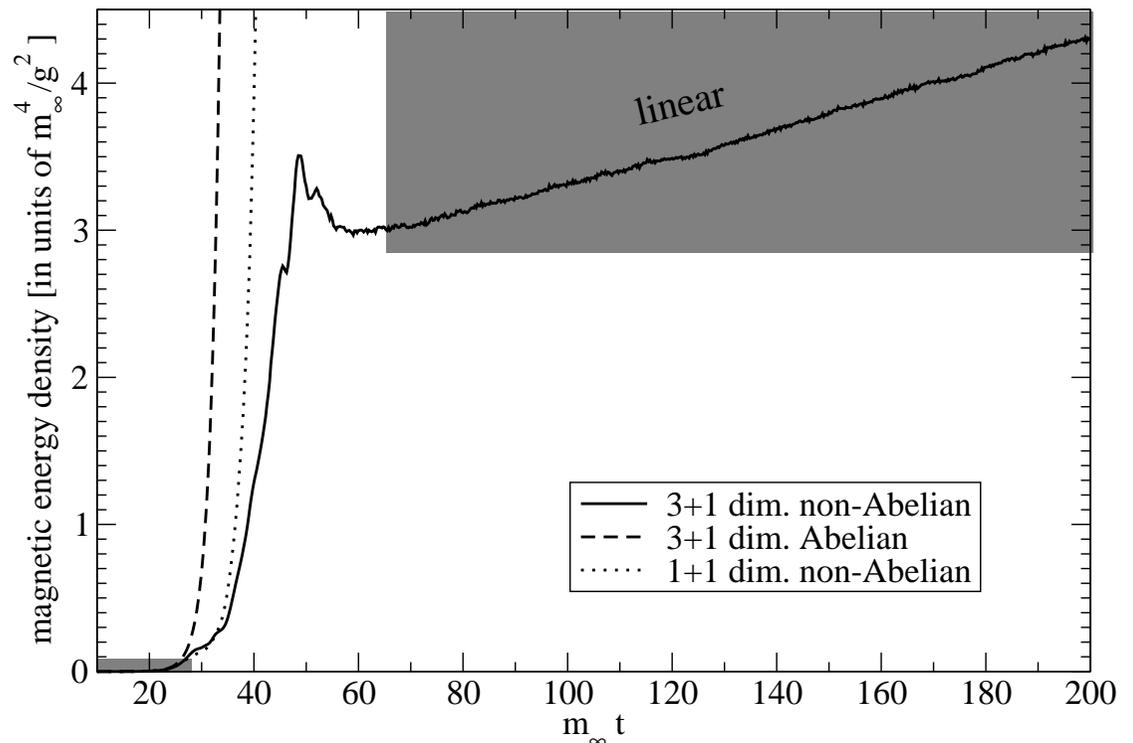
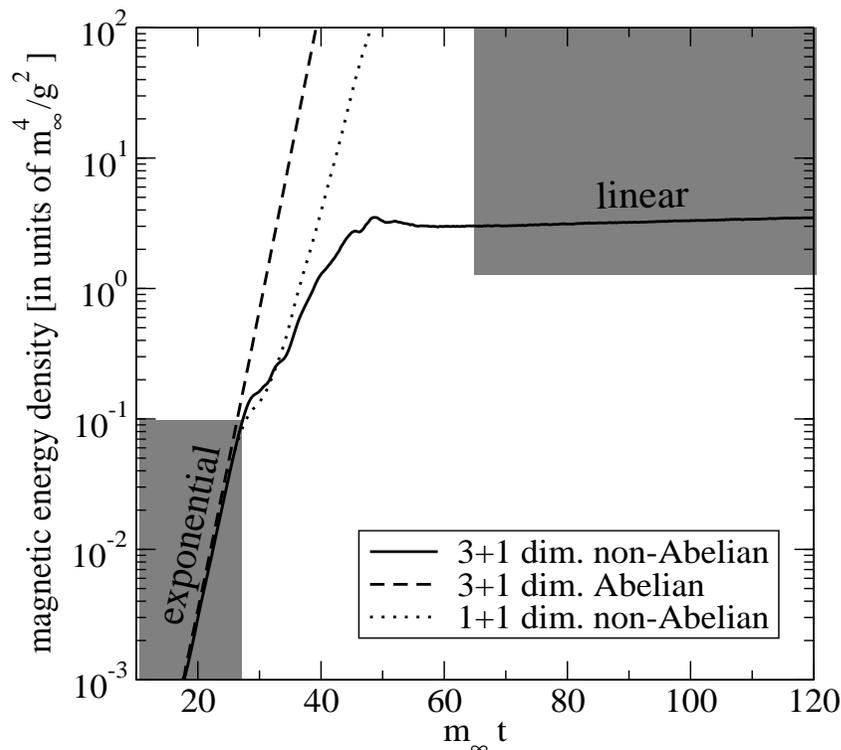
However: local Abelianization can be destroyed by interactions with not perfectly transversely constant modes

3D+3V

However: local Abelianization can be destroyed by interactions with not perfectly transversely constant modes

→ attenuation of exponential growth to only linear one:

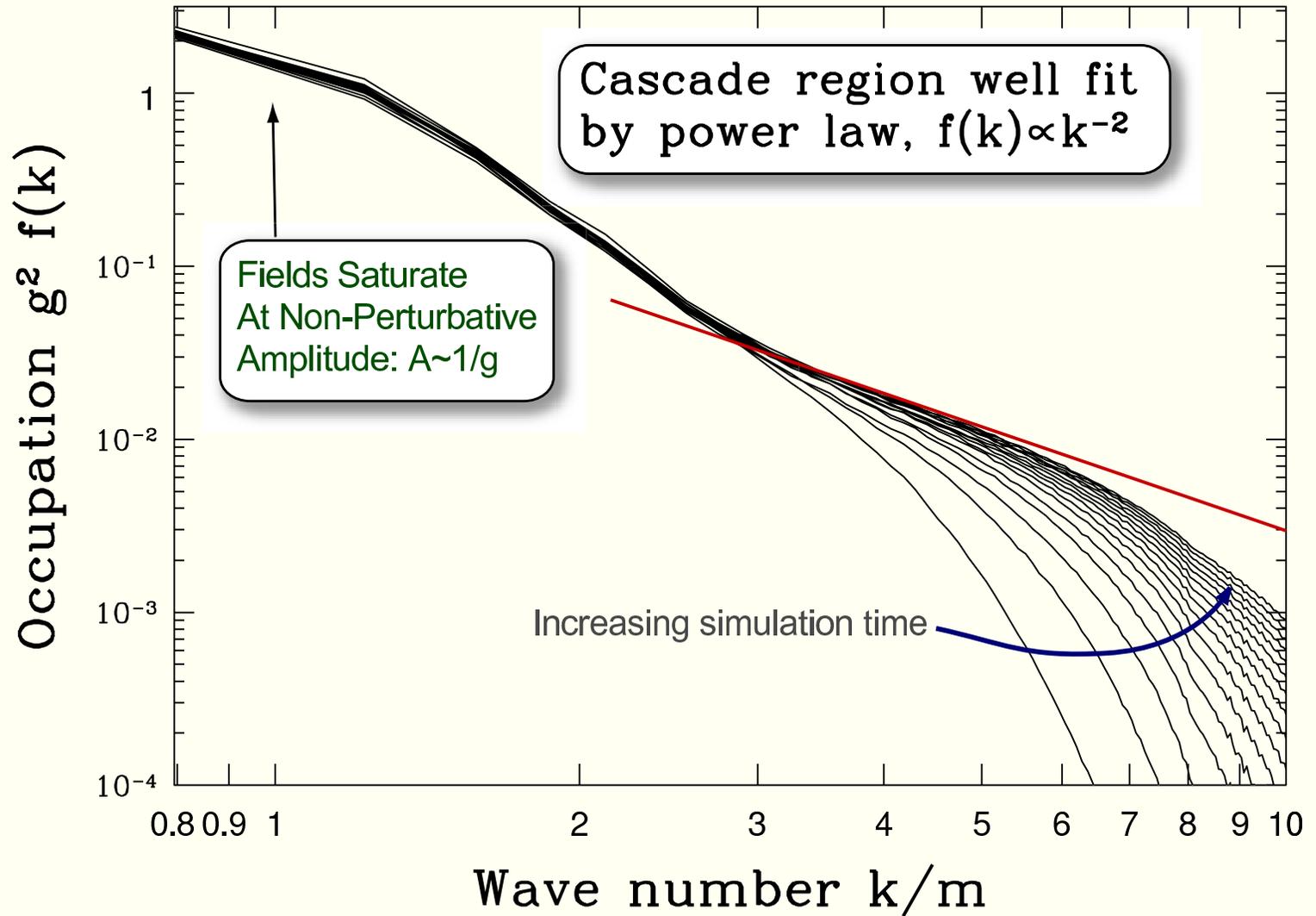
[Arnold, Moore & Yaffe, PRD72 ('05) 054003]



same in discoball discretization: [AR, Romatschke & Strickland, JHEP 09 \(2005\) 041](#)

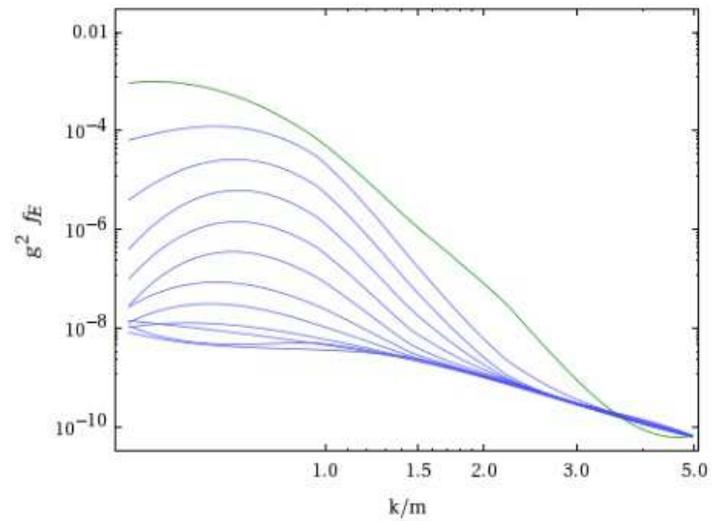
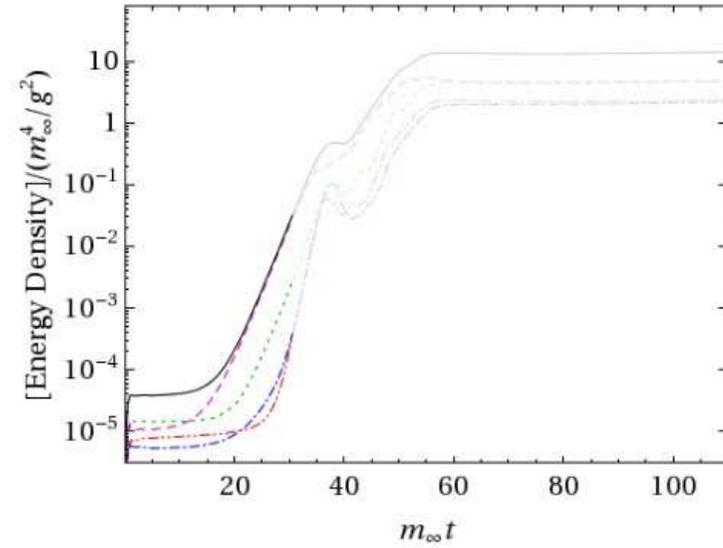
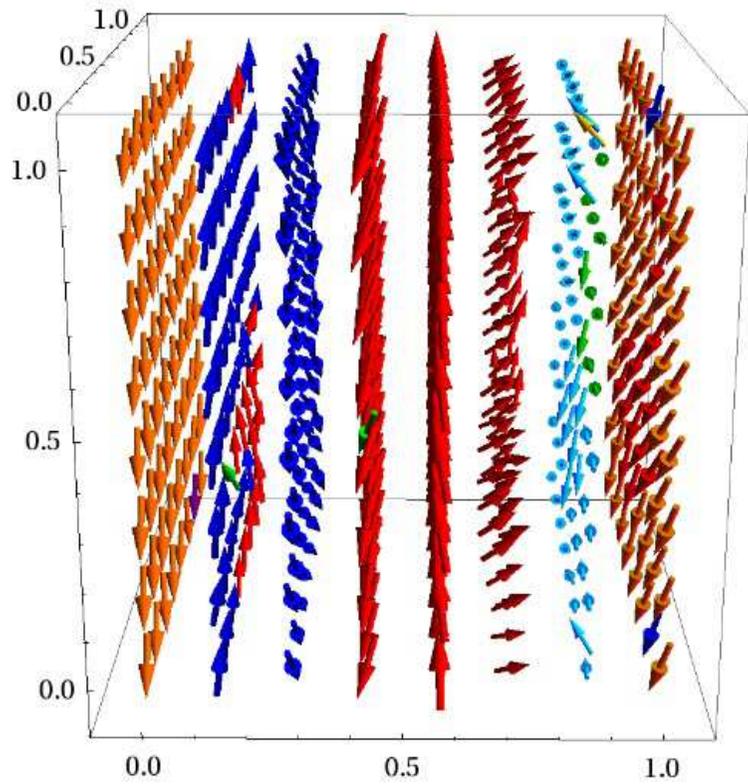
larger lattices, larger anisotropies: [Bödeker & Rummukainen, JHEP 07 \(2007\) 02](#)

Cascade



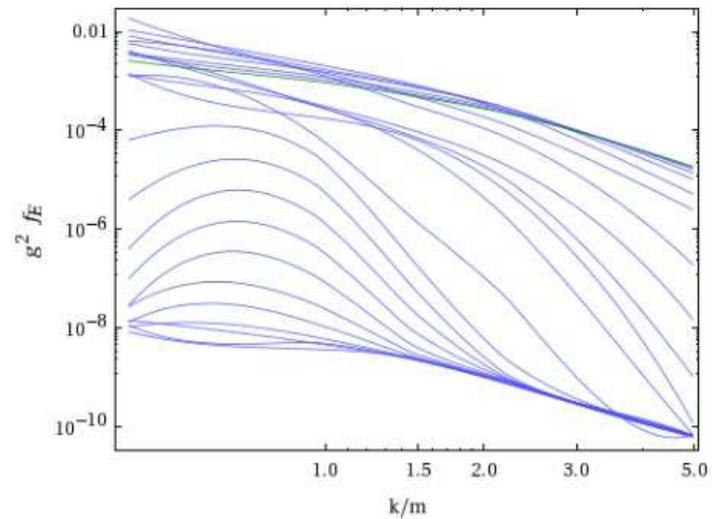
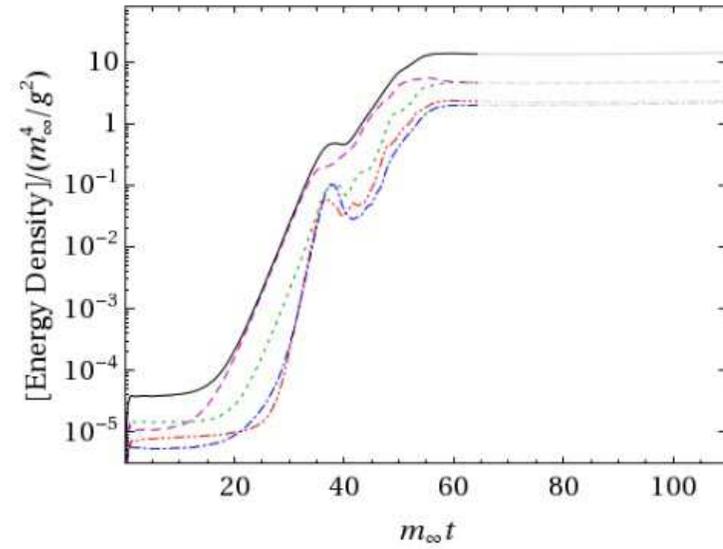
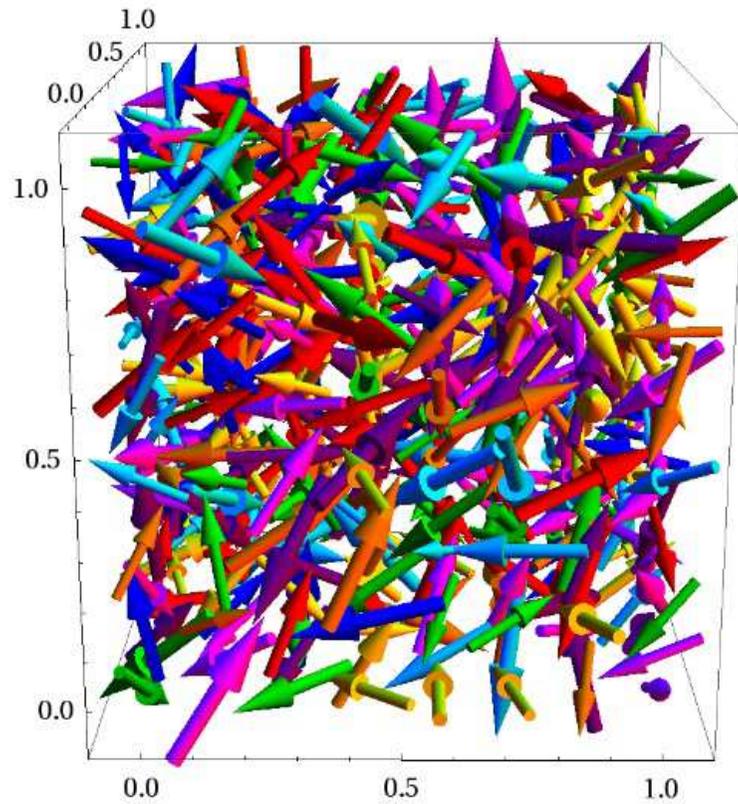
3D+3V and SU(3)

A. Ipp, AR, M. Strickland, to appear



3D+3V and SU(3)

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Plasma instabilities in Bjorken expansion

Longitudinal (Bjorken) expansion: Competition between

- increasing anisotropy (more and more modes become unstable)
- and decreasing density (\leftrightarrow growth rate)

[Romatschke & AR, PRL 97 (2006)]

Plasma instabilities in Bjorken expansion

Longitudinal (Bjorken) expansion: Competition between

- increasing anisotropy (more and more modes become unstable)
- and decreasing density (\leftrightarrow growth rate)

[Romatschke & AR, PRL 97 (2006)]

Notation: proper time $\tau = \sqrt{t^2 - z^2}$ and space-time rapidity $\eta = \text{atanh} \frac{z}{t}$

$$x^\mu \rightarrow x^\alpha = (\tau, x^i, \eta) \text{ with } g_{\alpha\beta} = (1, -1, -1, -\tau^2)$$

momentum rapidity $y = \text{atanh} \frac{p^0}{p^z}$:

$$p^\mu \rightarrow p^\alpha = |\mathbf{p}_\perp| (\cosh(y - \eta), \cos \phi, \sin \phi, \underbrace{\tau^{-1} \sinh(y - \eta)}_{p'^z / |\mathbf{p}_\perp|})$$

Boost-invariant free-streaming background e.g.

$$f_0(\mathbf{p}, x) = f_{\text{iso}} \left(\sqrt{p_\perp^2 + p_\eta^2 / \tau_{\text{iso}}^2} \right) = f_{\text{iso}} \left(\sqrt{p_\perp^2 + (p'^z \tau / \tau_{\text{iso}})^2} \right)$$

with space-time dependent anisotropy parameter $\xi(\tau) = (\tau / \tau_{\text{iso}})^2 - 1$

Hard-Expanding-Loop formalism

$$v \cdot D W_\alpha(\tau, x^i, \eta; \phi, y)|_{\phi, y} = v^\beta F_{\alpha\beta},$$

$$\frac{1}{\tau} D_\alpha [\tau g^{\alpha\gamma}(\tau) g^{\beta\delta}(\tau) F_{\gamma\delta}] = j^\beta,$$

where for $f_0(\mathbf{p}, x) = f_{\text{iso}} \left(\sqrt{p_\perp^2 + p_\eta^2 / \tau_{\text{iso}}^2} \right)$

$$j^\alpha(\tau, x^i, \eta) = -\frac{m_D^2(\tau = \tau_{\text{iso}})}{2} \int_0^{2\pi} \frac{d\phi}{2\pi} \int_{-\infty}^{\infty} dy v^\alpha \left(1 + \frac{\tau^2}{\tau_{\text{iso}}^2} \sinh^2(y - \eta) \right)^{-2} \\ \times \left\{ \cos \phi W_1 + \sin \phi W_2 - \frac{\tau}{\tau_{\text{iso}}^2} \sinh(y - \eta) W_\eta \right\}$$

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discretized HEL:

now *disco cylinders*: $\mathcal{W}(\tau, x^i, \eta; \phi, y)$ with equally spaced ϕ_n, y_m

AR, M. Strickland, M. Attems: PRD78 (2008)

Non-Abelian Discretized HEL

Hard gluon number density and initial fluctuation spectrum from **CGC** →

Parameters from saturation scenario $\tau_0 \simeq Q_s^{-1}$:

$$n(\tau_0) = c \frac{(N_c^2 - 1) Q_s^3}{4\pi^2 N_c \alpha_s (Q_s \tau_0)}$$

$$\text{with gluon liberation factor } c = \begin{cases} 0.5 & \text{Krasnitz '99 et al.} \rightarrow 1.1 \text{ Lappi '07 (numerical)} \\ \underline{2 \ln 2} \approx 1.39 & \text{Kovchegov (analytical estimate)} \end{cases}$$

$$f_{\text{iso}} = \mathcal{N} f_{\text{thermal}} \text{ with (transverse) temperature } T = 0.47 Q_s \text{ [Krasnitz et al.]}$$

$$\text{pure glue} \quad \rightarrow \quad \mathcal{N} = \frac{1}{\alpha_s} \frac{c}{8N_c (0.47)^3 \zeta(3)} \frac{\tau_0}{\tau_{\text{iso}}} \frac{1}{Q_s \tau_0}$$

$$\rightarrow \quad \frac{\mu}{Q_s} = \frac{1}{8} m_D^2 \pi \tau_{\text{iso}} Q_s^{-1} = \frac{\pi^2}{48 \cdot 0.47 \cdot \zeta(3)} c \approx \begin{cases} 0.182 & (c = 0.5) \\ \underline{0.505} & (c = 2 \ln 2) \end{cases}$$

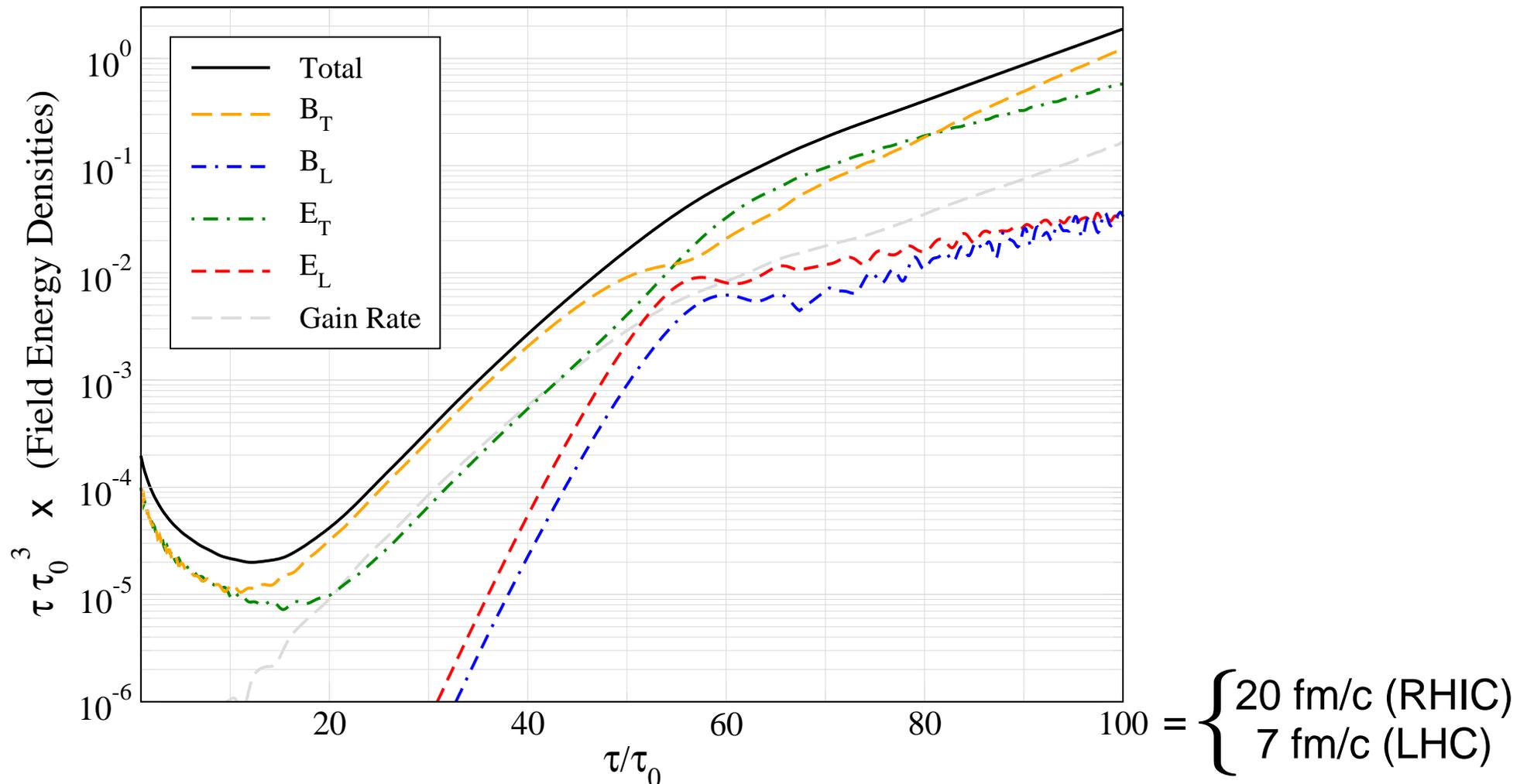
$$Q_s \simeq 1 \text{ GeV (RHIC)} \dots 3 \text{ GeV (LHC) ?}$$

+ form of initial fluctuation spectrum from Fukushima, McLerran & Gelis 2007

Non-Abelian Discretized HEL – Chromofield energy densities

Hard gluon number density and initial fluctuation spectrum from **CGC** →

Uncomfortably long delay of growth to $\tau/\tau_0 \simeq 15 \leftrightarrow \begin{cases} 3 \text{ fm/c (RHIC)} \\ 1 \text{ fm/c (LHC)} \end{cases}$



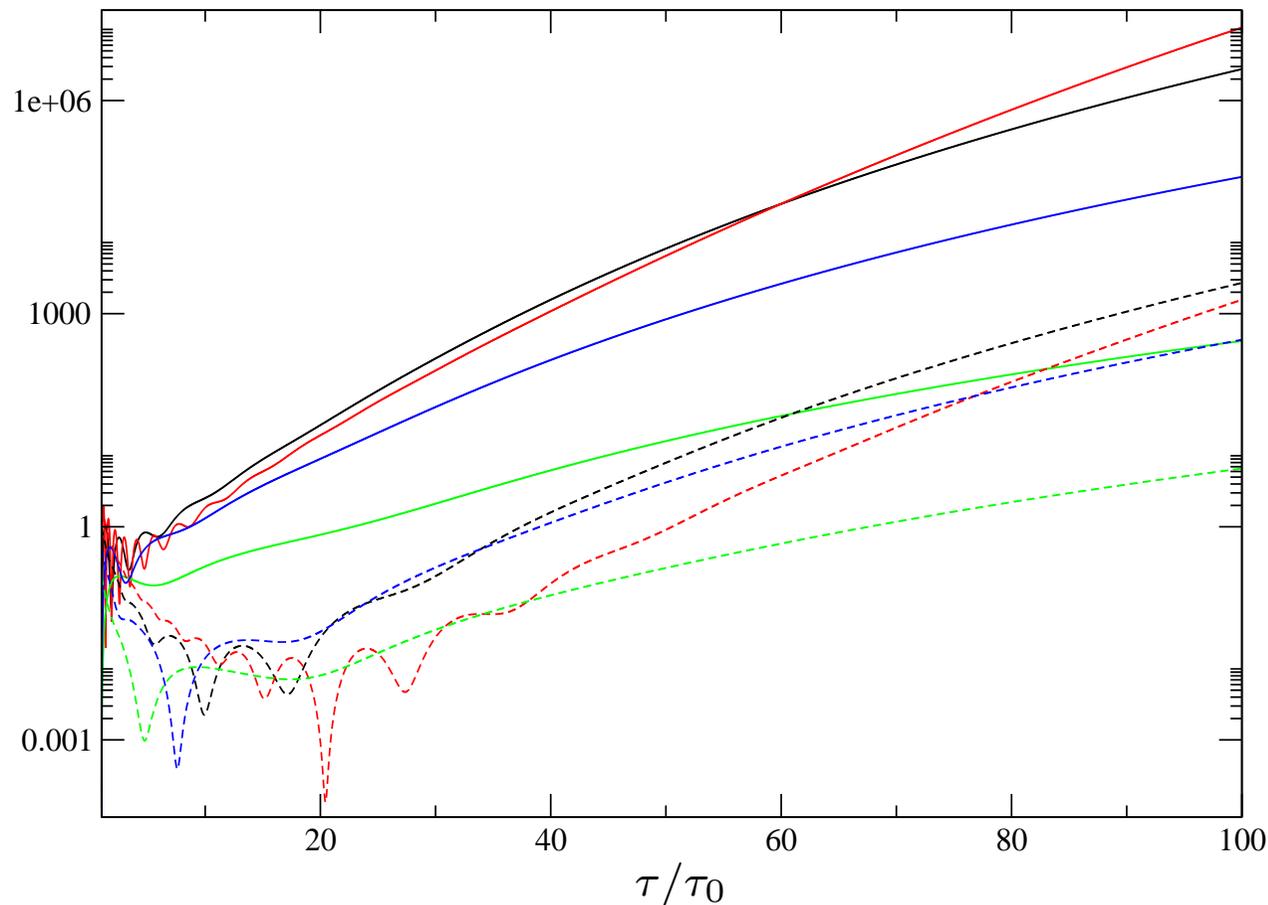
Quick Rise of Plasma Instabilities

[AR & D. Steineder, arXiv:0912.5383]

generalization to initial fluctuations in W fields (hard particle distribution)

→ **reduction of delay by almost one magnitude!**

Longitudinal wave vector (most unstable, effectively 1+1 modes)



= $\begin{cases} 20 \text{ fm/c (RHIC)} \\ 7 \text{ fm/c (LHC)} \end{cases}$

3D+3V HEL

Work in progress: full 3D+3V nonabelian HEL

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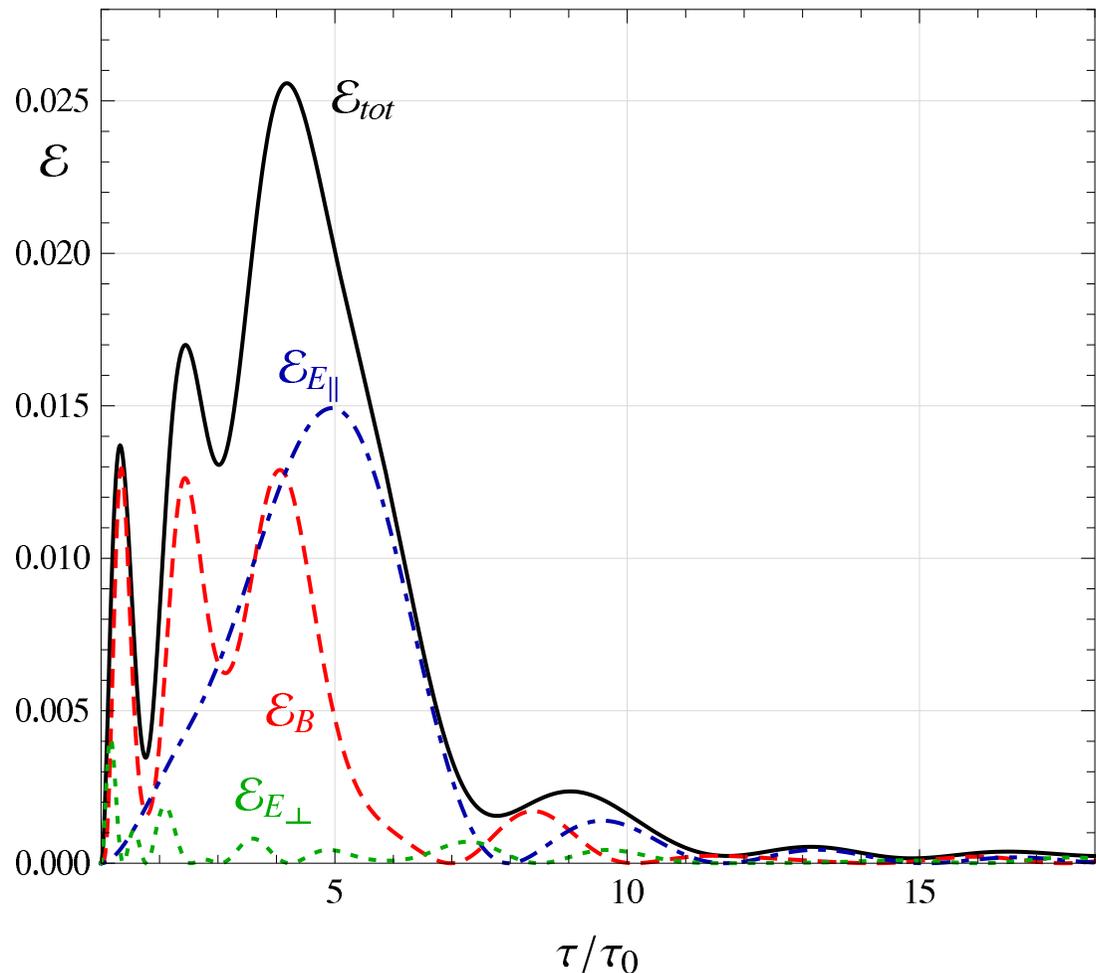
Just completed: full semi-analytic study of linear (effectively abelian) regime

[AR & D. Steineder, arXiv:0912.5383]

E.g.: electric (Buneman) instability for wave vector not parallel to z -axis but initially within

45° cone to the z -axis:

(turned off by expansion because wave vector rotates away from z axis)



Conclusions and Outlook

Conclusions:

- Plasma instabilities are parametrically dominant phenomenon in anisotropic plasmas with interesting characteristic time scales
- Uncomfortably long delay of onset of plasma instabilities in Bjorken expansion now understood to be due to having had fluctuations only in gluon fields — virtually absent when including also fluctuations in hard particle distribution
- More likely important role for plasma instabilities at LHC!

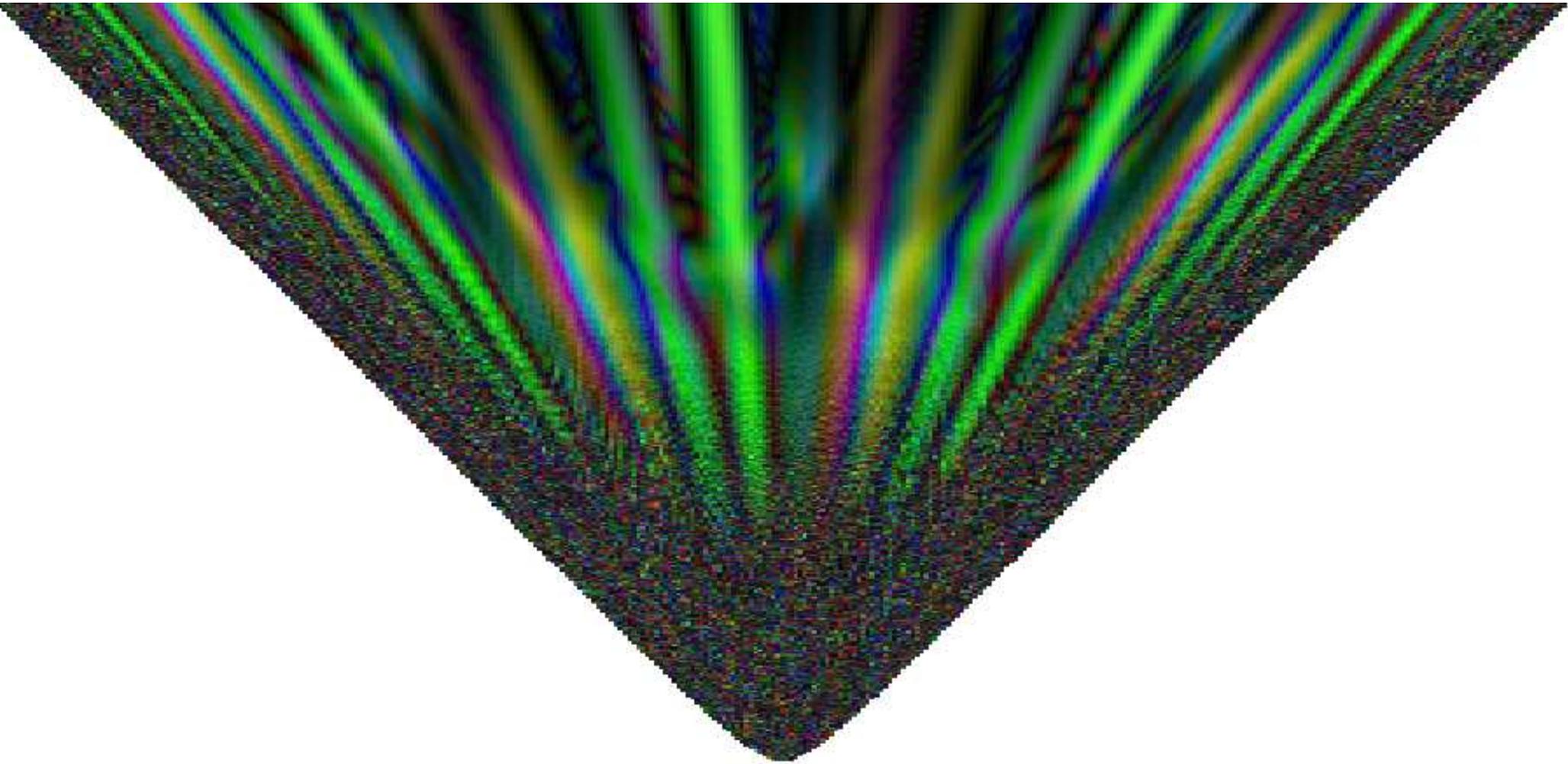
Forthcoming:

- Full 3D+3V
- needed for analysis of turbulent regime in expanding case

Open challenge:

- Complete perturbative bottom-up thermalization scenario

Non-Abelian Discretized HEL — Visualization in Lab Frame



Supplement: Transversely constant modes in linear (Abelian) regime

Most unstable modes for $\tau > \tau_{\text{iso}}$ have $\partial_i A^\alpha \equiv 0$

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Linearize ($A^\tau = 0$): $\left[\frac{1}{\tau} \partial_\tau \tau \partial_\tau - \frac{1}{\tau^2} \partial_\eta^2 \right] A^i(\tau, \eta) = j^i, \quad \partial_\tau \frac{1}{\tau} \partial_\tau A_\eta = \frac{j_\eta}{\tau},$

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Solving $v \cdot \partial W = v^\beta F_{\alpha\beta}$:

$$W_\alpha(\tau, \eta; \phi, y) = \int_{\tau_0}^\tau d\tau' \frac{v^\beta F_{\alpha\beta}|_{\tau', \eta(\tau')}}{\cosh(y - \eta(\tau'))}, \quad y - \eta(\tau') = \text{asinh}\left(\frac{\tau}{\tau'} \sinh(y - \eta)\right),$$

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$$\begin{aligned} \longrightarrow j^i[W] &= -\frac{m_D^2}{4} \int_{-\infty}^{\infty} dy \left(1 + \frac{v_\eta^2}{\tau_{\text{iso}}^2}\right)^{-2} \int_{\tau_0}^\tau d\tau' \\ &\quad \times \left[\left(\partial'_\tau - \frac{\tanh \bar{\eta}'}{\tau'} \partial_{\eta'} \right) A^i(\tau', \eta') + \frac{v_\eta}{\tau_{\text{iso}}^2} \frac{\partial_{\eta'} A^i(\tau', \eta')}{\cosh \bar{\eta}'} \right], \end{aligned}$$

$$j^\eta[W] = -\frac{m_D^2}{2\tau_{\text{iso}}^2} \int \frac{dy v^\eta v_\eta}{\left(1 + \frac{v_\eta^2}{\tau_{\text{iso}}^2}\right)^2} \int_{\tau_0}^\tau d\tau' \partial_{\tau'} A_\eta(\tau', \eta'),$$

where $\eta' = \eta(\tau')$ and $\bar{\eta}' = \eta(\tau') - y$.

Transversely constant modes in linear (Abelian) regime

Fourier transform in space-time rapidity ($\nu \sim k_z \tau$ at $\eta \sim 0$)

$$A^i(\tau, \eta) = \int \frac{d\nu}{2\pi} \exp(i\nu\eta) \tilde{A}^i(\tau, \nu),$$

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(similar equation for $\tilde{j}^\eta(\tau, \nu)$)

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(similar equation for $\tilde{j}^\eta(\tau, \nu)$)

Integro-differential equations, solved by numerical leap-frog algorithm

$$\begin{aligned} \tau \partial_\tau \tilde{A}^i(\tau, \nu) &= \tilde{\Pi}^i(\tau, \nu) \quad \text{and} \\ \partial_\tau \tilde{\Pi}^i(\tau, \nu) &= -\nu^2 \tau^{-1} \tilde{A}^i(\tau, \nu) + \tau \tilde{j}^i(\tau, \nu) \end{aligned}$$

Transversely constant modes in linear regime: *Analytical results*

Late-time behavior: approximate 4th order ODE

$$\tau \gg \tau_0 \gtrsim \tau_{\text{iso}}: \left[\partial_\tau^2 \tau \partial_\tau \tau \partial_\tau + \nu^2 \partial_\tau^2 + \mu \partial_\tau^2 \tau - \mu \nu^2 \frac{1}{\tau} \right] \tilde{A}^i(\tau, \nu) \approx 0,$$
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Stable plasma oscillations for $\nu \ll 1$:

$$\tilde{A}^i(\tau, \nu) = c_1 J_0(2\sqrt{\mu\tau}) + c_2 Y_0(2\sqrt{\mu\tau}),$$
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[Romatschke & Strickland, PRD68]

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[Romatschke & Strickland, PRD68]

Unstable transverse modes for $\nu \gtrsim 1$:

$$\tilde{A}^i(\tau, \nu) \sim \tau {}_2F_3 \left(\frac{3-\sqrt{1+4\nu^2}}{2}, \frac{3+\sqrt{1+4\nu^2}}{2}; 2, 2 - i\nu, 2 + i\nu; -\mu\tau \right)$$

$$\rightarrow \tau^{1/4} \exp(2\sqrt{\mu\tau}) \quad \text{for } \nu \gg 1$$

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$$\left[\partial_\tau \frac{1}{\tau} \partial_\tau + \mu \frac{2}{\tau^2} \right] \tilde{A}_\eta(\tau, \nu) \approx 0, \text{ where } \boxed{\mu = \frac{1}{8} m_D^2 \pi \tau_{\text{iso}}}.$$

Stable plasma oscillations for $\nu \ll 1$:

$$\tilde{A}^i(\tau, \nu) = c_1 J_0(2\sqrt{\mu\tau}) + c_2 Y_0(2\sqrt{\mu\tau}),$$

$$\tau^{-1} \tilde{A}_\eta(\tau, \nu) = c_1 J_2(2\sqrt{2\mu\tau}) + c_2 Y_2(2\sqrt{2\mu\tau}), \text{ indeed: } \lim_{\xi \rightarrow \infty} \omega_{\text{pl}}^\ell / \omega_{\text{pl}}^t = \sqrt{2}$$

[Romatschke & Strickland, PRD68]

Unstable transverse modes for $\nu \gtrsim 1$:

$$\tilde{A}^i(\tau, \nu) \sim \tau {}_2F_3 \left(\frac{3-\sqrt{1+4\nu^2}}{2}, \frac{3+\sqrt{1+4\nu^2}}{2}; 2, 2 - i\nu, 2 + i\nu; -\mu\tau \right)$$

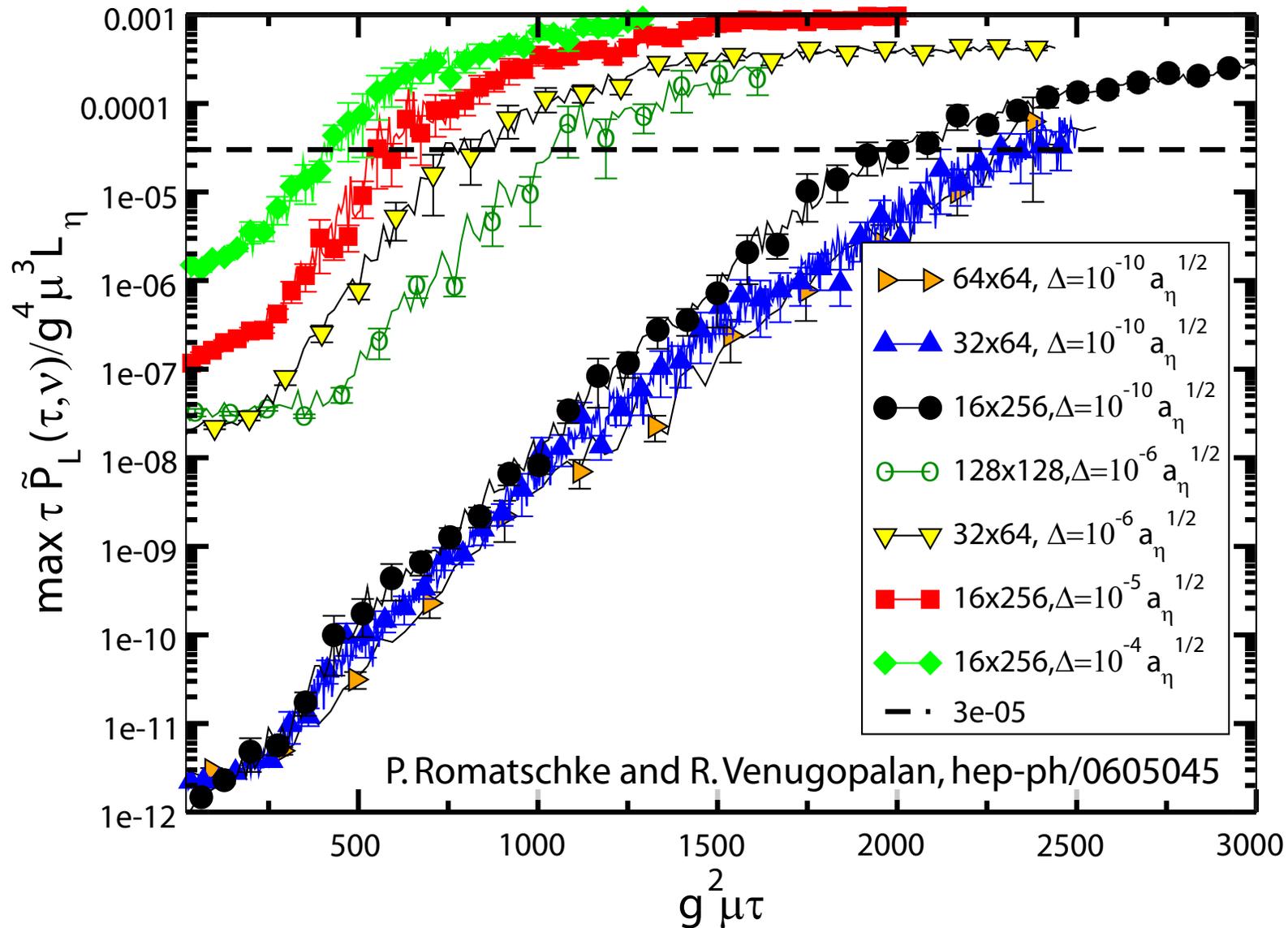
$$\rightarrow \tau^{1/4} \exp(2\sqrt{\mu\tau}) \quad \text{for } \nu \gg 1$$

qualitative agreement with unstable melting color glass-condensate

of [P. Romatschke & R. Venugopalan, PRL96(2006)062302; hep-ph/0605045]

Unstable glasma

P. Romatschke and R. Venugopalan, PRL 96, PRD 74 (2006)



Transversely constant modes in linear regime: *Numerical results*

Numerical result vs. asymptotic ${}_2F_3$ behavior (thin bright lines) ($c = 0.5$)

