NTNU
Faculty for Science and Technology
Department of Physics

Exam TFY 4210 Applied Quantum Mechanics 2010

Lecturer: Professor Jens O. Andersen
Department of Physics Phone: 73593131

Saturday June 12 2010
09.00-13.00h

Examination support:
Approved calculator
Rottmann: Matematisk Formelsamling
Rottmann: Matematische Formelsammlung
Barnett & Cronin: Mathematical Formulae

The metric is $(1, -1, -1, -1)$ and the units are $\hbar = c = 1$. The problem set is four pages. Useful formulas are listed at the end. Read carefully. Viel Glück!

Problem 1

The Lagrangian for a complex scalar field with mass m and charge q coupled to an electromagnetic field is

$$\mathcal{L} = (D_\mu \Phi)^* (D^\mu \Phi) - m^2 \Phi^* \Phi - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} .$$

1) Show that the equation of motion for Φ^* is

$$\left[D_\mu D^\mu + m^2 \right] \Phi = 0 .$$

In the following we will consider a single particle in two spatial dimensions with charge q in an external field given by the vector potential $A^\mu = (0, -By, 0, 0)$, where B is a constant.

2) Calculate the electric and magnetic fields $\mathbf{E}$ and $\mathbf{B}$.

3) Show that the Hamiltonian density can be written as

$$\begin{aligned}\mathcal{H} &= (\partial_0\Phi)^* (\partial_0\Phi) + [(\partial_x - iqBy)\Phi^*] [(\partial_x + iqBy)\Phi] + (\partial_y\Phi)^* (\partial_y\Phi) \\ &\quad + m^2\Phi^*\Phi + \frac{1}{2}B^2 .\end{aligned}$$

Is $\mathcal{H}$ Lorentz invariant?

4) The eigenfunctions of the equation of motion can be written as as

$$\Phi = e^{-i(Et - p_x x)} f(y) ,$$

where E is the energy of the state and p_x is the x-component of the momentum. Show by substition into the equation of motion that $f(y)$ satisfies

$$\left[-\frac{d^2}{dy^2} - E^2 + m^2 + (p_x - qBy)^2\right] f(y) = 0 .$$

Use this to compute the spectrum, i.e. find the energy eigenvalues E. Hint: Harmonic oscillator.

Problem 2

Consider a dilute Bose gas at zero temperature. Let $a_\mathbf{p}$ be the annihilation operator for a particle with momentum $\mathbf{p}$ and $a^\dagger_\mathbf{p}$ be a creation operator for a particle with momentum $\mathbf{p}$. The Bogolibov transformation is given by

$$\begin{aligned}a^\dagger_\mathbf{p} &= u_\mathbf{p} b^\dagger_\mathbf{p} + v_{-\mathbf{p}} b_{-\mathbf{p}} \\ a_\mathbf{p} &= u_\mathbf{p} b_\mathbf{p} + v_{-\mathbf{p}} b^\dagger_{-\mathbf{p}} .\end{aligned}$$

Note that the coefficients $u_\mathbf{p}$ and $v_{-\mathbf{p}}$ are real.

1) Find the relation that $u_\mathbf{p}$ and $v_{-\mathbf{p}}$ must satisfy if we demand that the quasiparticle operators $b_\mathbf{p}$ and $b^\dagger_\mathbf{p}$ satisfy the standard commutation relations

$$[b_\mathbf{p}, b^\dagger_\mathbf{k}] = \delta_{\mathbf{p},\mathbf{k}} ,$$

and all other commutators vanish.

2) The interacting ground state of the Bose gas is denoted by $|\Phi\rangle$. Calculate the average

$$\langle a_{\mathbf{p}} a^\dagger_{\mathbf{p}} \rangle \;=\; \langle \Phi | a_{\mathbf{p}} a^\dagger_{\mathbf{p}} | \Phi \rangle \ .$$

3) Explain briefly *condensate depletion* in the context of Bose-Einstein condensation.

Problem 3

Consider a weakly interacting Bose gas with action

$$S \;=\; \int dt\, d^3x \left[i\psi^\dagger(\mathbf{x},t)\partial_0\psi(\mathbf{x},t) + \mu\psi^\dagger(\mathbf{x},t)\psi(\mathbf{x},t) - \frac{1}{2m}\nabla\psi^\dagger(\mathbf{x},t)\cdot\nabla\psi(\mathbf{x},t) - \frac{1}{2}g\left(\psi^\dagger(\mathbf{x},t)\psi(\mathbf{x},t)\right)^2 \right] \ ,$$

where ψ is a complex bosonic field, μ is the chemical potential, and $g = 4\pi a/m$, where m is the mass of the boson and a is the scattering length. We write the complex field ψ in polar form

$$\psi(\mathbf{x},t) = \sqrt{\sigma(\mathbf{x},t)}e^{i\phi(\mathbf{x},t)} \ ,$$

where σ is the density operator $\psi^\dagger\psi$ and ϕ is the phase of ψ. At $T = 0$, the system is in a Bose-Einstein condensed phase and we denote by ρ_0 the magntitude of the condensate. We must then replace σ by $\rho_0 + \tilde\sigma$, where $\tilde\sigma$ is a quantum fluctuating field. Inserting the polar parametrization of ψ into the action, one finds

$$S \;=\; \int dt\, d^3x \left\{ -(\rho_0+\tilde\sigma)\partial_0\phi - \frac{1}{2m}\left[(\rho_0+\tilde\sigma)(\nabla\phi)^2 + \frac{(\nabla\tilde\sigma)^2}{4(\rho_0+\tilde\sigma)}\right] + \mu(\rho_0+\tilde\sigma) - \frac{1}{2}g(\rho_0+\tilde\sigma)^2 \right\} \ ,$$

where we have omitted a total divergence. In the following, we assume that density fluctuations are small compared to ρ_0 and so we can make a Taylor expansion of $1/4(\rho_0+\tilde\sigma)$ in the action. This yields

$$S \;=\; \int dt\, d^3x \left\{ -(\rho_0+\tilde\sigma)\partial_0\phi - \frac{1}{2m}\left[(\rho_0+\tilde\sigma)(\nabla\phi)^2 + \frac{(\nabla\tilde\sigma)^2}{4\rho_0} + ...\right] + \mu(\rho_0+\tilde\sigma) - \frac{1}{2}g(\rho_0+\tilde\sigma)^2 \right\} \ ,$$

where the dots indicate terms that are higher-order $\tilde\sigma$.

1) Show that the equation of motion for $\tilde\sigma$ can be written as

$$\tilde\sigma \;=\; -\frac{1}{g}\left[\partial_0\phi + \frac{1}{2m}(\nabla\phi)^2 - A\nabla^2\tilde\sigma\right] \ .$$

and determine the coefficient A. Hint: Use $\mu = g\rho_0$.

2) The equation of motion for $\tilde{\sigma}$ can be solved by iteration. As a first approximation we therefore set $A = 0$. Use its equation of motion to eliminate $\tilde{\sigma}$ from the action and show that we can write

$$S = \int dt\, d^3x \left\{ B \left[\partial_0 \phi + \frac{1}{2m} (\nabla \phi)^2 \right] + C \left[\partial_0 \phi + \frac{1}{2m} (\nabla \phi)^2 \right]^2 + ... \right\} ,$$

and determine the coefficients B and C.

3) A Galilean transformation is defined by

$$\begin{aligned} \mathbf{x}' &= \mathbf{x} - \mathbf{v}t , \\ t' &= t , \end{aligned}$$

where $\mathbf{v}$ is the velocity of the frame S' with respect to the frame S. The various derivatives of the phase transform as

$$\begin{aligned} \nabla \phi &\rightarrow \nabla \phi + m\mathbf{v} , \\ \partial_0 \phi &\rightarrow \partial_0 \phi - \mathbf{v} \cdot \nabla \phi - \frac{1}{2} m v^2 . \end{aligned}$$

Is the action for the phase invariant under Galilean transformations?

4) Find the propagator for the field ϕ.

5) Find the spectrum and comment on the result.

Useful formulas

$$\begin{aligned} \mathcal{H}_{\text{Maxwell}} &= \frac{1}{2} (\mathbf{E}^2 + \mathbf{B}^2) , \\ D_\mu &= \partial_\mu + iqA_\mu , \\ F_{\mu\nu} &= \partial_\mu A_\nu - \partial_\nu A_\mu . \end{aligned}$$