FY3452 - Solutions Exercise set 8 spring 2016

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Problem 15.10

The metric is

$$ds^2 = -\left(1-\frac{r^2}{R^2}\right)dt^2 + \left(1-\frac{r^2}{R^2}\right)^{-1}dr^2 + r^2d\Omega^2\ . \tag{1}$$

We would like to introduce a new time coordinate $\tilde{t}$ such that the metric will be of the form

$$ds^2 = -(1-f)\,d\tilde{t}^2 + (1+f)dr^2 - 2f\,dr\,d\tilde{t} + r^2d\Omega^2\ , \tag{2}$$

where $f = r^2/R^2$. The idea to this ansatz did not arise by divine inspiration but from a textbook on general relativity. You can also solve the problem using the same steps as in the case of the Schwarzschild metric. The transformation is written as

$$t = \tilde{t} + g(r)\ , \tag{3}$$

where $g(r)$ is an unknown function. Inserting $dt = \tilde{dt} + g'(r)dr$ into the line element (1), we find

$$ds^2 = -(1-f)\,d\tilde{t}^2 - 2g'(r)(1-f)d\tilde{t}dr - [g'(r)]^2(1-f)dr^2 + \frac{dr^2}{1-f} + r^2d\Omega^2\ . \tag{4}$$

Comparing this equation with the ansatz, we find

$$g'(r) = \frac{f}{1-f}\ . \tag{5}$$

We do not need the explicit form of $g(r)$. The radial null lines satisfy

$$-(1-f)\,d\tilde{t}^2 - 2f\,dr\,d\tilde{t} + (1+f)dr^2 = 0\ . \tag{6}$$

One possibility is $dr = d\tilde{t}$ or

$$\tilde{t} = r + \text{constant}\ . \tag{7}$$

This corresponds to *outgoing* light rays in a $(r, \tilde{t})$ diagram.

The other light rays are found as follows. Dividing by dr^2 and rearranging terms, we obtain

$$(1-f)\left(\frac{d\tilde{t}}{dr} - \frac{f}{1-f}\right)^2 = \frac{1}{1-f} \,. \tag{8}$$

Rearranging and taking the square root, we find

$$\begin{aligned}\frac{d\tilde{t}}{dr} &= \pm\frac{1+f}{1-f} \\ &= \pm\frac{1+r^2/R^2}{1-r^2/R^2} \,.\end{aligned} \tag{9}$$

For $r > R$, the solution is

$$\tilde{t} = \pm\left[-r + 2R\,\mathrm{arctanh}\left(\frac{R}{r}\right)\right] \,. \tag{10}$$

For $r < R$, the solution is

$$\tilde{t} = \pm\left[-r + 2R\,\mathrm{arctanh}\left(\frac{r}{R}\right)\right] \,. \tag{11}$$

These solutions diverge as $r \to R^{\pm}$ and do not cross $r = R$. Once $r > R$, we cannot enter the region $r < R$ since the only light ray crossing $r = R$ is outgoing.

Comment: The metric (1) is that of the socalled de Sitter space, which has a cosmological constant.

Problem 15.14

The four-velocity of an observer rotating with angular velocity Ω is

$$\mathbf{u}_{\mathrm{obs}} = u^t_{\mathrm{obs}}(1, 0, 0, \Omega)$$

We next use the normalization of the four-velocity. This yields

$$\mathbf{u}_{\mathrm{obs}} \cdot \mathbf{u}_{\mathrm{obs}} = (u^t_{\mathrm{obs}})^2\left[g_{tt} + 2\Omega g_{t\phi} + \Omega^2 g_{\phi\phi}\right] = -1 \,. \tag{12}$$

The quantity inside the brackets of (12) must be negative. The range of the angular velocity Ω is then given by the zeros of $g_{tt} + 2\Omega g_{t\phi} + \Omega^2 g_{\phi\phi}$. These are denoted by $\Omega_{\pm}$ and given by

$$\Omega_{\pm} = -\frac{g_{t\phi}}{g_{\phi\phi}} \pm \sqrt{\left(\frac{g_{t\phi}}{g_{\phi\phi}}\right)^2 - \left(\frac{g_{tt}}{g_{\phi\phi}}\right)} \,. \tag{13}$$

Thus the allowed angular velocities satisfy

$$\underline{\underline{\Omega_- \leq \Omega \leq \Omega_+}} \,. \tag{14}$$

As $r \to r_+$, we find

$$g_{tt} \to \frac{a^2 \sin^2\theta}{\rho_+^2} \, , \qquad g_{\phi\phi} \to \frac{(2Mr_+)^2}{\rho_+^2} \sin^2\theta \, , \tag{15}$$

where $\rho_+ = r_+^2 + a^2 \sin^2\theta$. Thus the two solutions coincide and the only allowed angular velocity is

$$\begin{aligned} \Omega_H &= -\left(\frac{g_{t\phi}}{g_{\phi\phi}}\right)_{r=r_+} \\ &= \underline{\underline{\frac{a}{2Mr_+}}} \, . \end{aligned} \tag{16}$$