TFY4305 Nonlinear Dynamics, Final Exam December 8, 2006 Solution Set

Problem 1

a) $\dot{x} = 0 \Rightarrow y = 0$ and $\dot{y} = 0 \Rightarrow x = 0$. $\lambda = (\mu \pm \sqrt{\mu^2 - 4})/2$. Real part $\mu/2$ vanishes as $\mu \to 0$.

b) Complex eigenvalues $\Rightarrow$ ocillatory solutions.

c) $\dot{r} = h_1(r, \theta) = r(\mu - r)\sin^2\theta$, $\dot{\theta} = h_2(r, \theta) = -1 + (\mu - r)\cos\theta\sin\theta$.

d) $\mu \ll 1$ and $r \ll 1 \Rightarrow \dot{\theta} = -1 \Rightarrow \theta(t) = -t + \theta_0$.

e) As $\theta(t) = -t+\theta_0$, $\langle\sin^2\theta\rangle = 1/2$ (see Strogatz, p. 224). This leads to $\dot{r} = h(r) = r(\mu - r)/2$. $\dot{r} = 0 \Rightarrow r^* = 0$ or $r^* = \mu$.

f) $h'(r) = \mu/2 - r \Rightarrow h'(r^* = 0) = \mu/2$ and $h'(r^* = \mu) = -\mu/2$.

g) $r^* = 0$ is stable for $\mu < 0$ and unstable for $\mu > 0$. $r^* = \mu$ is unstable for $\mu < 0$ and stable for $\mu > 0$.

h) The bifurcation is *not* a *generic* Hopf bifurcation as the radius of the limit cycle grows as μ and not as $\sqrt{\mu}$, see Strogatz, p. 251.

Problem 2

a) $f_r(x_n^*) = x_n^* \Rightarrow x_n^* = 0$, $x_n^* = +\sqrt{1-r}$ and $x_n^* = -\sqrt{1-r}$.

b) $f_r'(x_n) = r + 3x_n^2$. $f_r'(x_n^* = 0) = r \Rightarrow x_n^* = 0$ is stable in the interval $[-1, 1]$, so that $r_- = -1$ and $r_+ = +1$.

c) $f_r'(x_n^* = \pm\sqrt{1-r}) = 3 - 2r$. Fixed points $x_n^* = \pm\sqrt{1-r}$ are always unstable (when $r \leq 1$ where they exist).

d) At $r = r_+ = 1$ there is a *pitchfork bifurcation* where an unstable fixed point bifurcates into two unstable fixed points and becomes stable itself.

e) At $r = r_- = -1$, a period doubling takes place for $x_n^* = 0$ (eigenvalue $= -1$).

f) $f_r^2(x^c) \approx (1 + 2\epsilon)x^c - (2 + 4\epsilon)(x^c)^3 = x^c$. It is a second order equation with solution $x^c \approx \pm\sqrt{\epsilon}$.