Chapter 1

Lorentz transformations

1.1 Boosts

Let S and S' be two inertial frames, where S' moves with speed v relative to S along the x-axis. Let (x, y, z, t) and (x', y', z', t') be the space-time coordinates in the two inertial frames. The clocks are synchronized such that the origins O of S coincides with the origin O' of S' for $t = t' = 0$.

The coordinates in the two coordinate systems are then related by the following transformations:

$$x' = \gamma(x - vt) , \tag{1.1}$$
$$y' = y , \tag{1.2}$$
$$z' = z , \tag{1.3}$$
$$t' = \gamma(t - vx/c^2) , \tag{1.4}$$

where $\gamma = 1/\sqrt{1 - v^2/c^2}$. This can conveniently be written in matrix form

$$\begin{pmatrix} x' \\ ct' \end{pmatrix} = \begin{pmatrix} \cosh\theta & -\sinh\theta \\ -\sinh\theta & \cosh\theta \end{pmatrix} \begin{pmatrix} x \\ ct \end{pmatrix} , \tag{1.5}$$

where $\cosh\theta = \gamma$ and $\sinh\theta = \gamma v/c$. This gives

$$\tanh\theta = \frac{v}{c} . \tag{1.6}$$

Note that coordinates x and t are both involved in the transformations (1.1)-(1.4). This has important consequences: observers in S and S' do not (necessarily) longer agree that two events are taking place at the same time. The idea of *simultaneous events* is not absolute. More specifically, it leads to the following important effects

- **Lorentz contraction**: A rod of length L at rest is shorter by a factor $1/\gamma$ when moving with speed v.

- **Time dilation**: A moving clock is slower than a clock at rest.

Since space and time are intertwined, we introduce the idea of space-time or Minkowski space with coordinates (x, y, z, ct). The points in space-time are *events*, i.e. "something happening" at time t at a specific point in space [1].

The determinant of matrix (1.5) is unity. This is the same as for a proper rotation matrix in $\mathcal{R}$:

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sinh\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} , \tag{1.7}$$

If we write $ct = iT$, the matrix (1.5) is transformed into the matrix (1.7) (Exercise!) Thus, a boost can be viewed as a rotation.

The Newtonian limit of (1.1)-(1.4) is found by setting $\gamma = 1$, i.e. by assuming $v \ll c$ and keeping only terms that are at most first order in v/c. This yields

$$x' = (x - vt) , \tag{1.8}$$
$$y' = y , \tag{1.9}$$
$$z' = z , \tag{1.10}$$
$$t' = t . \tag{1.11}$$

This transformation is known as a *Galilean transformation* and is the nonrelativistic limit of special relativity. Note in particular that idea of simultaneity is absolute.

1.2 Geometry of Minkowski space

We next introduce the distance $(\Delta s)^2$ between the two points in Minkowski space by

$$(\Delta s)^2 = c^2(\Delta t)^2 - \left[(\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2\right] . \tag{1.12}$$

In $\mathcal{S}'$, we obtain

$$\begin{aligned} (\Delta s')^2 &= c^2(\Delta t')^2 - \left[(\Delta x')^2 + (\Delta y')^2 + (\Delta z')^2\right] \\ &= \gamma^2 c^2 \left[(\Delta t)^2 - \frac{2v\Delta x\Delta}{c^2} + \frac{v^2}{c^4}(\Delta x)^2\right] - \gamma^2\left[(\Delta x)^2 - v^2(\Delta t)^2 - 2v\Delta x\Delta t\right] \\ &\quad -(\Delta y)^2 - (\Delta z)^2 \\ &= \gamma^2 c^2(\Delta t)^2\left[1 - \frac{v^2}{c^2}\right] - \gamma^2(\Delta x)^2\left[1 - \frac{v^2}{c^2}\right] - (\Delta y)^2 - (\Delta z)^2 \\ &= c^2(\Delta t)^2 - \left[(\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2\right] , \end{aligned} \tag{1.13}$$

[1] An example of event is the emission of light at time t by a source located at (x, y, z).

where we in the last line have used that $\gamma = 1/\sqrt{1 - v^2/c^2}$. Thus the quantity $(\Delta s)^2$ is independent of the inertial frame. In fact, one can show that the quantity is invariant under all Lorentz transformations and it can therefore be viewed as a *geometric* quantity independent of coordinate systems. In differential form, we write it as

$$ds^2 \;=\; c^2dt^2 - dx^2 - dy^2 - dz^2 \;. \tag{1.14}$$

We next define $x^\mu = (ct, \mathbf{x})$. In other words, $x^0 = ct$, $x^1 = x$, $x^2 = y$, and $x^3 = z$, and so $\mu = 0, 1, 2, 3$. Moreover, we introduce the metric tensor, $g_{\mu\nu}$ which can be written as a 4×4 matrix:

$$g_{\mu\nu} \;=\; \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & -1 \end{pmatrix} . \tag{1.15}$$

This is also in shorthand notation: $g = \mathrm{diag}(1, -1, -1, -1)$. We can then write

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu \;, \tag{1.16}$$

where we are using Einstein summation convention: One is summing over repeated indices (one index upstairs and one index downstairs).

We next introduce the concept of a *contravariant* vector, which is four quantities A^μ that transform like x^μ under Lorentz transformations. Moreover, we define a *covariant* vector A_μ in terms of the metric tensor and the contravariant vector A^ν by

$$A_\mu \;=\; g_{\mu\nu}A^\nu \tag{1.17}$$

For example, the covariant vector $x_\mu = (ct, -\mathbf{x})$. The metric can now be written as a product between a contravariant and a covariant vector:

$$ds^2 \;=\; dx_\mu dx^\mu \;. \tag{1.18}$$

We have seen that ds^2 is the same in all inertial frames, i,.e it is scalar.

1.2.1 Differential Operators

We next discuss how the differential operator ∂/∂_μ transforms under Lorentz transformations. We restrict ourselves to boosts along the x-axis. Using the chain rule, we can write

$$\frac{\partial}{\partial x'} \;=\; \frac{\partial x}{\partial x'}\frac{\partial}{\partial x} + \frac{\partial t}{\partial x'}\frac{\partial}{\partial t} \tag{1.19}$$

$$\frac{\partial}{\partial y'} \;=\; \frac{\partial}{\partial y} \tag{1.20}$$

$$\frac{\partial t}{\partial z'} \;=\; \frac{\partial}{\partial z} \tag{1.21}$$

$$\frac{\partial}{\partial t'} \;=\; \frac{\partial x}{\partial t'}\frac{\partial}{\partial x} + \frac{\partial t}{\partial t'}\frac{\partial}{\partial t} \;. \tag{1.22}$$

In order to find the partial derivatives, we need the inverse transformations of () given by

$$x = \gamma(x' + vt') , \quad (1.23)$$
$$y = y' , \quad (1.24)$$
$$z = z' , \quad (1.25)$$
$$t = \gamma(t' + vx') . \quad (1.26)$$

This yields

$$\frac{\partial}{\partial x'} = \gamma\left[\frac{\partial}{\partial x} + v\frac{\partial}{\partial t}\right] \quad (1.27)$$
$$\frac{\partial}{\partial y'} = \frac{\partial}{\partial y} \quad (1.28)$$
$$\frac{\partial t}{\partial z'} = \frac{\partial}{\partial z} \quad (1.29)$$
$$\frac{\partial}{\partial t'} = \gamma\left[\frac{\partial}{\partial t} + v\frac{\partial}{\partial x}\right] . \quad (1.30)$$

If we compare with the transformation rules for $x_\mu = (-\mathbf{x}, t)$

$$-x' = \gamma[-x - vt] , \quad (1.31)$$
$$-y' = -y , \quad (1.32)$$
$$-z' = -z , \quad (1.33)$$
$$t' = \gamma[t + vx] , \quad (1.34)$$

we conclude that the differential operator $\partial/\partial x_\mu$ transforms as a *covariant vector*. We denote it therefore by ∂_μ.

Exercise

Show that $\partial/\partial x^\mu$ transforms as a contravariant vector under Lorentz transformations.

From the above, we conclude that the operator

$$\partial_\mu \partial^\mu = g_{\mu\nu}\partial^\mu\partial^\nu , \quad (1.35)$$

is a *scalar* under Lorentz transformations.

Consider a rotation in the xy-plane by an angle θ. We can represent a rotation in the plane by a 2×2 matrix which relates the new and old coordinates:

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} . \quad (1.36)$$

This yields

$$\begin{aligned}
\frac{\partial}{\partial x'} &= \frac{\partial x}{\partial x'}\frac{\partial}{\partial x} + \frac{\partial y}{\partial x'}\frac{\partial}{\partial y} \\
&= \cos\theta\frac{\partial}{\partial x} + \sin\theta\frac{\partial}{\partial y} \qquad (1.37)\\
\frac{\partial}{\partial y'} &= \frac{\partial x}{\partial y'}\frac{\partial}{\partial x} + \frac{\partial y}{\partial y'}\frac{\partial}{\partial y} \\
&= -\sin\theta\frac{\partial}{\partial x} + \cos\theta\frac{\partial}{\partial y}\ . \qquad (1.38)
\end{aligned}$$

The operator ∇^2 is then invariant under rotation:

$$\begin{aligned}
(\nabla')^2 &= \left(\frac{\partial}{\partial x'}\right)^2 + \left(\frac{\partial}{\partial y'}\right)^2 \\
& \left(\frac{\partial}{\partial x}\right)^2 + \left(\frac{\partial}{\partial y}\right)^2 \\
&= \nabla^2\ . \qquad (1.39)
\end{aligned}$$

This result can be easily generalized to three spatial dimensions. In that case the rotation matrix is a 3×3 matrix and is parameterized by three angles, the so-called Euler angles.

Bibliography

[1] J. O. Andersen, *Introduction to Statistical Mechanics*, NTNU-trykk third edition 2009.